

# CONFIGURATION SPACE INTEGRALS AND THE COHOMOLOGY OF THE SPACE OF HOMOTOPY STRING LINKS

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**ABSTRACT.** Configuration space integrals have in recent years been used for studying the cohomology of spaces of (string) knots and links in  $\mathbb{R}^n$  for  $n > 3$  since they provide a map from a certain differential algebra of diagrams to the deRham complex of differential forms on the spaces of knots and links. We refine this construction so that it now applies to the space of homotopy string links – the space of smooth maps of some number of copies of  $\mathbb{R}$  in  $\mathbb{R}^n$  with fixed behavior outside a compact set and such that the images of the copies of  $\mathbb{R}$  are disjoint – even for  $n = 3$ . We further study the case  $n = 3$  in degree zero and show that our integrals represent a universal finite type invariant of the space of classical homotopy string links. As a consequence, we obtain configuration space integral expressions for Milnor invariants of string links.

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## 1. INTRODUCTION

This paper is concerned with the study of the cohomology of the space of homotopy string links (or long homotopy links)  $\mathcal{H}_m^n$  using configuration space integrals, also known as Bott-Taubes integrals. This is the space of smooth maps of  $m$  copies of  $\mathbb{R}$  in  $\mathbb{R}^n$  where the images of the various copies of  $\mathbb{R}$  are disjoint and where the map is fixed outside some compact set (see Definition 2.3). Our main results are

- (i) For  $m \geq 1$  and  $n \geq 3$ , there exists a certain differential algebra of diagrams  $\mathcal{HD}$  and a differential algebra map

$$(1) \quad I_{\mathcal{H}}: \mathcal{HD} \longrightarrow \Omega^*(\mathcal{H}_m^n),$$

where  $\Omega^*$  stands for the deRham complex of differential forms (Theorem 4.28).

- (ii) In degree zero and for  $n = 3$ , this map produces all finite type invariants of homotopy string links (Theorem 5.8).
- (iii) As a consequence of the previous result, we obtain configuration space integral expressions for Milnor invariants of homotopy string links in  $\mathbb{R}^3$  (Theorem 5.14).

The first two results parallel those for string knots  $\mathcal{K}^n$ , i.e. embeddings of  $\mathbb{R}$  in  $\mathbb{R}^n$  [4, 5, 6, 25], and more generally string links  $\mathcal{L}_m^n$ , i.e. embeddings of  $m$  copies of  $\mathbb{R}$  in  $\mathbb{R}^n$  [27], where all maps are always prescribed outside some compact set. At the same time, these results are also very different. To explain, we first briefly review the standard construction of the map

$$(2) \quad \bar{I}_{\mathcal{L}}: \mathcal{LD} \longrightarrow \Omega^*(\mathcal{L}_m^n)$$

that corresponds to the one in (1) and is familiar from the literature [5, 27]. In particular,  $\mathcal{LD}$  is a familiar diagram complex associated to the space of string links.

To produce forms on  $\mathcal{L}_m^n$ , one first creates fiber bundles of configuration spaces over this space. The bundle depends on a diagram in  $\mathcal{LD}$ . A diagram has vertices that abstractly represent configurations of points on and off a link, and its edges prescribe a way to pull back copies of the volume  $(n-1)$ -form from the sphere  $S^{n-1}$  to the total space of the bundle. We then integrate this pullback form along the fiber, thereby producing a form on  $\mathcal{L}_m^n$ . One of the main reasons this construction works is that ordinary embedded links behave well with respect to restriction, i.e. the restriction map for links is a fibration by the Isotopy Extension Theorem.

The situation is different for  $\mathcal{H}_m^n$  because homotopy links are not embeddings and the restriction map is far from a fibration (see Section 4.2.2). Thus the obvious generalization of the above fails to extend to  $\mathcal{H}_m^n$ . The main contribution of this paper is a refinement of the construction of the fiber bundles which makes it possible to integrate over  $\mathcal{H}_m^n$ . The short explanation of this refinement is that, in the construction of  $\bar{I}_{\mathcal{L}}$ , only vertices of the diagram determine the bundle, while in our construction, both vertices and edges are relevant. This leads to breaking up the diagram according to its “grafts” (see Definition 4.8 and Definition 4.11) and the construction of what is essentially a product bundle over the set of graft components. In this fashion we

construct a new map

$$(3) \quad I_{\mathcal{L}}: \mathcal{LD} \longrightarrow \Omega^*(\mathcal{L}_m^n),$$

identify a subcomplex  $\mathcal{HD} \subset \mathcal{LD}$ , exhibit the map from equation (1), and show that the diagram

$$\begin{array}{ccc} \mathcal{HD} & \xrightarrow{\quad} & \mathcal{LD} \\ I_{\mathcal{H}} \downarrow & & \downarrow I_{\mathcal{L}} \\ \Omega^*(\mathcal{H}_m^n) & \longrightarrow & \Omega^*(\mathcal{L}_m^n) \end{array}$$

commutes. After we define  $I_{\mathcal{L}}$  and show how it restricts to the map  $I_{\mathcal{H}}$ , we show in Proposition 4.20 that the old integration map  $\bar{I}_{\mathcal{L}}$  and our map  $I_{\mathcal{L}}$  produce the same form. Thus our construction is indeed a refinement of the one considered by others.

One interesting attribute of our construction of  $I_{\mathcal{H}}$  is that this map can be defined even when  $n = 3$ , which is not the case with  $I_{\mathcal{L}}$ . The reason is that the issue of the vanishing of the integration along a certain part of the boundary of the bundle, the so-called *anomalous face*, is not present for homotopy links (see Remark 5.1). This is potentially an exciting feature since it brings the map  $I_{\mathcal{H}}$  closer to being a quasi-isomorphism for  $n = 3$ ; this is already widely believed for  $\bar{I}_{\mathcal{L}}$  (and hence  $I_{\mathcal{L}}$ ) for  $n > 3$ , but now we can conjecture the same for  $I_{\mathcal{H}}$  and  $n \geq 3$ .

The anomalous face also makes an appearance in the study of finite type invariants of knots and links via configuration space integrals [25, 26, 27]. As stated in (ii) above, we extend this study to the case of homotopy string links. The difference is that, for (string) knots and links, these integrals represent a universal finite type invariant only up to an indeterminacy due to the non-vanishing of anomalous faces (see Section 5.1). However, this is not a problem for homotopy string links and we in Theorem 5.8 give the correspondence between weight systems (functionals on diagrams in degree zero satisfying some relations) and finite type invariants of homotopy string links without any indeterminacy.

Theorem 5.8 connects to other work that has been done on finite type invariants of homotopy string links. To show that  $I_{\mathcal{H}}$  represents the universal finite type invariant of homotopy string links, we first show that the zeroth cohomology of the complex  $\mathcal{HD}$  gives a certain vector space of diagrams that has already been studied [3, 14, 16, 27]. Our construction, however, is dictated by geometry – we have arrived at  $\mathcal{HD}$  by looking for spaces we could integrate over to get forms on  $\mathcal{H}_m^n$ . Further, we are concerned with all  $n \geq 3$ , and for  $n = 3$  and degree zero we happen to have obtained the “correct” diagrams and relations. This means that our approach is indeed a generalization, with a new perspective, of existing work.

Since Milnor invariants of homotopy string links are known to be finite type, Theorem 5.8 immediately gives an entirely novel configuration space integral construction for Milnor invariants (as mentioned in (iii)). Further, some connections between tree diagrams and Milnor invariants arise naturally from our construction and the authors plan to pursue this in a future paper [20]. More details about the planned work on Milnor invariants are given in Section 5.4.

The philosophy in this paper is thus to reconstruct all the ingredients of the map (3), but in an improved and refined fashion, and then show at every important instance of the construction how everything works when one restricts to the case of homotopy string links. Consequently, we have had to be precise and detailed about the definition and structures in the diagram complex

$\mathcal{LD}$ , the fiber bundles mentioned earlier, the degree zero case, etc. This has required us to fill in some of the details that have been missing from the literature. Some instances of this are:

- $\mathcal{LD}$  is now defined purely combinatorially (it had largely been done through pictures before, and mainly for the case of knots);
- the correspondence between the shuffle product on  $\mathcal{LD}$  and the wedge product on  $\Omega^*(\mathcal{L}_m^n)$  has been elucidated;
- the appearance of the STU and IHX relations in degree zero has been treated thoroughly;
- essentially all the details of the proof that configuration space integrals represent a universal finite type invariant of embedded string links and homotopy string links are given (the most complete proof for knots is in [26], but only an outline for links is given in [27]);

In addition, the work here unifies and extends many seemingly disparate results in the subject of configuration space integrals (case  $n > 3$  is in literature usually treated separately from the case  $n = 3$ ). All of this makes for a self-contained and thorough treatment of how configuration space integrals are used in knot and link theory. We hope that in addition to establishing some new and useful results, this paper will serve as a practical and a beneficial introduction to the subject.

Finally, it is worth noting where the results from this paper fit into the larger program of the authors. The overarching goal is the study of homotopy string links (and embedded string links) in the context of manifold calculus of functors. To that end, the authors have developed its multivariable version [23], as well a cosimplicial model for the functor calculus Taylor tower for homotopy string links [21]. Using this model, it will be shown in [22] that the map  $I_{\mathcal{L}}$  factors through the Taylor tower and that this tower classifies finite type invariants. The goal is to use this to reprove the Habegger-Lin classification of homotopy links [9] as well as to try to extend some of their results to ordinary links. Along the way, the authors plan to study Milnor invariants in the context of manifold calculus using [20], which continues the exploration of the connection between configuration space integrals and Milnor invariants, as well as [19], which connects manifold calculus of certain generalizations of homotopy links to generalizations of Milnor invariants.

### 1.1. Organization of the paper.

- In Section 2, we define the spaces of string links and homotopy string links, make some observations about them, and set some notation and conventions.
- In Section 3, we define the diagram complex  $\mathcal{LD}$  and its subcomplex  $\mathcal{HD}$ . Section 3.1 contains the detailed definition of  $\mathcal{LD}$  and Section 3.2 discusses the differential and the shuffle product on this graded vector space. The subcomplex  $\mathcal{HD}$  is identified in Section 3.3. In Section 3.4, we show that  $\mathcal{LD}^0$  and  $\mathcal{HD}^0$  consist of trivalent diagrams modulo STU and IHX relations, plus an extra relation for  $\mathcal{HD}^0$  (Proposition 3.29). In that section we also describe the correspondence between trivalent and chord diagrams (Theorem 3.33). Many examples are given throughout.
- In Section 4, we construct the map  $I_{\mathcal{L}}$  in several steps. After reminding the reader about compactifications of configuration spaces in Section 4.1, we first recall in Section 4.2.1 the standard way of building a bundle of compactified configuration spaces over the

space of string links from a diagram  $\Gamma \in \mathcal{LD}$ . In Section 4.2.2, we show why this procedure fails to give bundles over the space of homotopy string links. Guided by how this procedure fails, we then go back to the complex  $\mathcal{LD}$ , define the graft components of a diagram in Section 4.2.3, and rework the definition of the bundle of configuration spaces based on these components in Section 4.2.4. The upshot is that these new bundles can now be defined over the space of links for any  $\Gamma \in \mathcal{LD}$  or over the space of homotopy links for any  $\Gamma \in \mathcal{HD} \subset \mathcal{LD}$ . In Section 4.3, we return to the main goal – producing forms on the space of (homotopy) links – and show how the edges of a diagram give a prescription for pulling back the product of volume forms to our bundles. Finally in Section 4.4 we describe how this pullback form can be pushed forward along the fiber of the bundle to  $\mathcal{L}_m^n$  or  $\mathcal{H}_m^n$  and give some examples. Proposition 4.20 states that the forms obtained using the standard definition of the bundles over  $\mathcal{L}_m^n$  and using our refined one are the same. This allows us to unify the old configuration space integral approach for string links with a new one for homotopy string links. What remains is to show that this integration is compatible with all the structure on  $\mathcal{LD}$ , and this is the content of Section 4.5 and Theorem 4.28 in particular.

- In Section 5, we study the case of classical homotopy string links ( $n = 3$ ) and prove that configuration space integrals represent a universal finite type invariant for this space (Theorem 5.8). We begin by discussing in Section 5.1 the anomalous face mentioned above and then review finite type theory and its connection to the combinatorics of chord diagrams in Section 5.2. Section 5.3 is finally devoted to the proof of Theorem 5.8. In Section 5.4, we deduce some quick consequences of Theorem 5.8 in regard to Milnor invariants. That section is meant to set the stage for the further study of Milnor invariants which the authors plan to undertake in [20, 22].

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## 2. SPACES OF STRING LINKS AND HOMOTOPY STRING LINKS

In this section, we define the spaces of string links and homotopy string links and set some conventions.

**Definition 2.1.** Let  $m \geq 1$ ,  $n \geq 2$  be integers. Let  $\sqcup_m \mathbb{R}$  denote the disjoint union of  $m$  copies of the real line, and let  $\text{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$  denote the space of smooth maps  $\sqcup_m \mathbb{R} \rightarrow \mathbb{R}^n$  which agree with the map which sends the  $i$ th copy of  $\mathbb{R}$  to  $(x, i, 0, 0, \dots, 0)$  outside some compact subset of  $\sqcup_m \mathbb{R}$ . This space is to be endowed with the  $\mathcal{C}^\infty$  topology.

The following is trivial.

**Proposition 2.2.**  $\text{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$  is contractible.

**Definition 2.3.**

- Let  $\mathcal{L}_m^n \subset \text{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$  denote the space of *string (or long) links in  $\mathbb{R}^n$  with  $m$  strands*. It consists of those maps  $L \in \text{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$  which are smooth embeddings (one-to-one maps whose derivatives are of maximal rank everywhere). A path in this space is called an *isotopy*.
- Let  $\mathcal{H}_m^n \subset \text{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$  denote the space of the space of *string (or long) homotopy links in  $\mathbb{R}^n$  with  $m$  strands*. It consists of those maps  $H \in \text{Map}_c(\sqcup_m \mathbb{R}, \mathbb{R}^n)$  such that if  $x$  and  $y$  are points in distinct copies of  $\mathbb{R}$ , then  $H(x) \neq H(y)$ . A path in this space is called a *link-homotopy*.

Note that  $\mathcal{H}_m^n$  is an example of a space of *link maps*, most recently studied in [8, 18, 19].

Throughout the paper, we will often drop the adjectives “string” and “long”, and refer to these objects as “links” and “homotopy links”. Each link and homotopy link is oriented in the sense that all copies of  $\mathbb{R}$  are given the usual orientation. The images of the copies of  $\mathbb{R}$  will be called *strands*.

The following corollary is immediate from Proposition 2.2.

**Corollary 2.4.**  $\mathcal{H}_1^n$  is contractible.

By  $\Omega^*(M)$  we mean the deRham cochain complex of differential forms on a manifold  $M$ . This is a differential algebra where the algebra structure is given by the wedge product of forms. The ground ring for all cohomology groups will be  $\mathbb{R}$ . In particular, in Section 5 we will be interested in  $H^0(\mathcal{H}_m^3)$ , i.e. the space of real-valued invariants of  $m$ -strand homotopy links in  $\mathbb{R}^3$ . The following observation will be useful there:

By general position, every homotopy link is link-homotopic to an embedded link. Moreover, by the remark following Definition 1.5 in [9], we can approximate a link-homotopy between embedded links by one which consists of isotopies and “crossing changes” of a strand with itself. A crossing change is a homotopy which takes place in the interior of a ball containing only two segments of a single strand, and the two segments cross during the homotopy. To check that something is an invariant of  $\mathcal{H}_m^3$ , it thus suffices to check that it is an invariant of  $\mathcal{L}_m^3$  and that it remains unchanged under such crossing changes. This observation will be used in the proof of Proposition 5.9 and will also allow us to connect the main results of Section 5 to Milnor invariants since these are in fact invariants of embedded links that are also invariant under crossing changes.

### 3. DIAGRAM COMPLEXES FOR THE SPACES OF STRING LINKS AND HOMOTOPY STRING LINKS

In this section we construct a diagram complex  $\mathcal{LD}$  and a subcomplex  $\mathcal{HD}$  that will serve as combinatorial prescriptions for producing cohomology classes on spaces of links and homotopy links. The complex  $\mathcal{LD}$  has been considered before [5, 27], but the definition of  $\mathcal{HD}$  appears to be new. As mentioned in the Introduction, in this section we also fill in some details in the definition and properties of  $\mathcal{LD}$ .

**3.1. Diagram complex for the space of string links.** While reading this section, the reader is encouraged to refer to Figure 1.

For a set  $S$ , we let  $SP_2(S)$  be the 2-fold symmetric product,

$$SP_2(S) = (S \times S) / \Sigma_2,$$

where  $\Sigma_2$  acts on the product  $S \times S$  by permuting the coordinates. We denote points in  $SP_2(S)$  as sets  $\{s_1, s_2\}$  where  $s_1, s_2 \in S$ , with the understanding that the cardinality of this set is one when  $s_1 = s_2$ .

**Definition 3.1.** A *diagram*  $\Gamma$  is a triple

$$\Gamma = (V(\Gamma), E(\Gamma), b_\Gamma)$$

where

- $V(\Gamma)$  is an ordered set called the *vertices* of  $\Gamma$ ;
- $E(\Gamma)$  is a set called the *edges* of  $\Gamma$ ; and
- $b_\Gamma : E(\Gamma) \rightarrow SP_2(V(\Gamma))$  is a map.

For an edge  $e \in E(\Gamma)$  with  $b(e) = \{v, w\}$ , we say that  $e$  *joins*  $v$  with  $w$ . When it is clear which diagram  $\Gamma$  we are speaking of we will write  $(V, E, b)$  in place of  $(V(\Gamma), E(\Gamma), b_\Gamma)$ .

The particular diagrams we study have a significant amount of extra structure. As we do not wish to impose cumbersome notation on the reader, we will continue to denote a diagram  $\Gamma$  with extra structure as a triple  $(V, E, b)$ , despite the possible ambiguity. Before describing the extra structures, we need some definitions and terminology.

**Definition 3.2.** For a diagram  $\Gamma = (V, E, b)$  and an edge  $e \in E$ , an *orientation* of  $e$  is a choice of injective map  $b(e) \rightarrow \{-1, 1\}$ .

Note that for an edge  $e$  such that  $b(e)$  consists of a single vertex, there are still two possible orientations, just as there are in the case where  $b(e)$  consists of two distinct vertices.

**Definition 3.3.** Let  $v, w$  be vertices in a diagram  $\Gamma = (V, E, b)$ . A *path* between  $v$  and  $w$  is a sequence  $\{e_i\}_{i=1}^k$  of edges  $e_i$  such that  $v \in b(e_1), w \in b(e_k)$  and  $b(e_i) \cap b(e_{i+1}) \neq \emptyset$  for all  $i$ .

Thus the orientations of edges, if they are present, are ignored for the purposes of defining a path.

One other definition we will have use for later is that of a connected component.

**Definition 3.4.** Let  $v$  be a vertex in a diagram  $\Gamma = (V, E, b)$ . The *connected component* of  $\Gamma$  *containing*  $v$  is the subdiagram  $(V', E', b')$ , where  $V'$  is the set of all vertices  $w$  that can be joined by a path of edges to  $v$ ,  $E'$  is the set of all edges that can appear in such paths, and  $b'$  is the restriction of  $b$ .

Fix integers  $m \geq 1$ ,  $n \geq 3$ , and let  $I_1, \dots, I_m$  be copies of the unit interval, which we will call *segments* for short. We think of  $\sqcup_i I_i$  as an ordered set according to the natural ordering of  $\{1, \dots, m\}$  and the natural ordering of  $I$ . Thus for  $x, y \in \sqcup_i I_i$ ,  $x \leq y$  whenever  $x \in I_i$  and  $y \in I_j$  and  $i < j$ , and when  $i = j$ ,  $x \leq y$  if this inequality holds under the usual ordering of  $I = [0, 1]$ .

**Definition 3.5.** Given integers  $m \geq 1$ ,  $n \geq 3$  as above, a *link diagram* is a diagram  $\Gamma = (V, E, b)$  together with the following extra structure. For the set  $V$  of vertices, we have



- A decomposition

$$V = V_{seg} \sqcup V_{free}$$

into ordered (possibly empty) sets, the elements of which are called *segment* and *free* vertices respectively.

- A decomposition of

$$V_{seg} = V_{seg,1} \sqcup \cdots \sqcup V_{seg,m}$$

into disjoint sets determined by the equivalence class of an injective function  $seg : V_{seg} \rightarrow \sqcup_i (I_i - \partial I_i)$  where  $V_{seg,k} = seg^{-1}(I_k - \partial I_k)$ , and which gives rise to the ordering of  $V_{seg}$  according to the ordering of  $\sqcup_i I_i$  described above. Two such injections  $s, s'$  are equivalent if they give rise to the same decomposition of  $V_{seg}$  and the same ordering on each of the sets in this decomposition according to the natural ordering of  $\sqcup_i I_i$ .

- An ordering of  $V$  given by the natural ordering of the ordered pair  $(V_{seg}, V_{free})$  of ordered sets.

For the set  $E$  of edges, we have a decomposition

$$E = E_{chord} \sqcup E_{mixed} \sqcup E_{free} \sqcup E_{loop}$$

into

- *chords*, joining distinct segment vertices;
- *mixed edges*, joining a free vertex with a segment vertex;
- *free edges*, joining distinct free vertices; and
- *loops*, joining a segment vertex with itself,

respectively. Moreover, each free vertex must have a path of edges to a segment vertex.

The *valence* of a vertex  $v$  is defined as follows. If  $v$  is a free vertex, its valence is the number of edges joining  $v$  to a vertex other than itself plus twice the number of loops joining  $v$  with itself. If  $v$  is a segment vertex, it is this sum plus two. The valence of each vertex in a link diagram is required to be at least three. In addition,

- If  $n$  is even, the set  $E$  of edges is ordered.
- If  $n$  is odd, each edge  $e \in E$  is oriented.

*Remark 3.6.* Another terminology for segment and free vertices is “external” and “internal”, respectively. This is because, in the case of knots, one has diagrams consisting of only one segment and if one is working with ordinary knots rather than long ones, the segment is drawn as a circle and free vertices are drawn inside it – hence “internal”. The vertices on the circle are then “external”. We decided that this terminology is misleading for our situation and prefer to call the vertices “segment” (those represented as lying on the  $m$  segments) and “free” (those not lying on the segments; these will later correspond to configuration points that are free to move in  $\mathbb{R}^n$ ).

We also distinguish “arcs” of a link diagram, which will be important when we define the differential, and should explain our seemingly strange definition of the valence of a segment vertex, as arcs contribute to the valence without counting as edges themselves.



**Definition 3.7.** For a link diagram  $\Gamma$ , an *arc* of  $\Gamma$  is a pair  $(v_1, v_2)$  of distinct segment vertices with  $v_1 < v_2$  whose images under the injection  $seg : V_{seg} \rightarrow \sqcup_i (I_i - \partial I_i)$  lie in the same segment, and such that the image of no other segment vertex lies between them.

We assume all possible arcs are present in any link diagram.

We pictorially represent a diagram in the plane with the intervals drawn as horizontal line segments, appearing in order from left to right and oriented from left to right, and each vertex as a point and each edge as an arc between vertices. Segment vertices are drawn on the intervals, and we think of arcs as segments in the intervals which lie between adjacent segment vertices. See Figure 1 below.

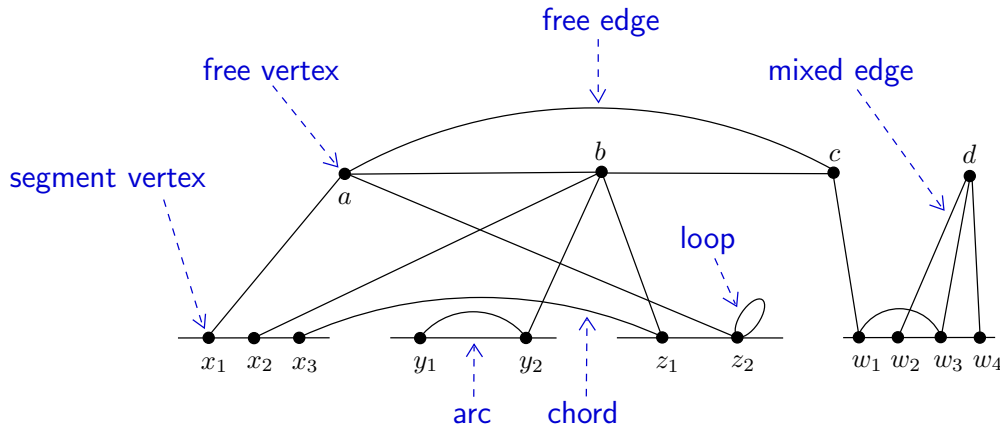


FIGURE 1. A diagram with four segments. Its edges may be labeled or oriented. Each vertex is at least trivalent (the valence of, say, vertex  $z_2$ , is five).

**Definition 3.8.** Link diagrams  $\Gamma = (V(\Gamma), E(\Gamma), b_\Gamma)$  and  $\Gamma' = (V(\Gamma'), E(\Gamma'), b_{\Gamma'})$  are *isomorphic* if there are order-preserving bijections  $\phi_V : V(\Gamma) \rightarrow V(\Gamma')$  and  $\phi_E : E(\Gamma) \rightarrow E(\Gamma')$  respecting the decomposition of the vertex set such that if  $\phi_V^* : SP_2(V(\Gamma)) \rightarrow SP_2(V(\Gamma'))$  denotes the induced map, then the diagram

$$\begin{array}{ccc} E(\Gamma) & \xrightarrow{b_\Gamma} & SP_2(V(\Gamma)) \\ \phi_E \downarrow & & \downarrow \phi_V^* \\ E(\Gamma') & \xrightarrow{b_{\Gamma'}} & SP_2(V(\Gamma')) \end{array}$$

commutes. In addition, if  $n$  is odd (so that each edge is oriented), for each edge  $e \in \Gamma$ , the injections  $o_{\Gamma, e} : b_\Gamma(e) \rightarrow \{-1, 1\}$  and  $o_{\Gamma', \phi_E(e)} : b_{\Gamma'}(\phi_E(e)) \rightarrow \{-1, 1\}$  must satisfy  $o_{\Gamma', \phi_E(e)} \circ \phi_V^* = o_{\Gamma, e}$ .

Note that “order-preserving” for edges is only relevant when the edge set is ordered.

**Definition 3.9.** The *degree* of a link diagram  $\Gamma = (V, E, b)$ , denoted  $\deg(\Gamma)$ , is defined to be

$$(4) \quad \deg(\Gamma) = 2|E| - 3|V_{free}| - |V_{seg}|.$$

Notice that, because we require the valence of each vertex in a link diagram to be at least three, the degree is nonnegative (there are no free vertices whose valence is less than three and there are no segment vertices whose valence is zero; if it were otherwise, the degree could be made arbitrarily negative). Thus we can think of the above degree as a measure of the failure of  $\Gamma$  to be trivalent. Indeed, when  $\deg(\Gamma) = 0$ , the segment and free vertices are precisely trivalent (in particular,  $\Gamma$  cannot contain loops). We will revisit such diagrams in Section 3.4.

**Definition 3.10.** When  $n$  is even (resp. odd), define  $\mathcal{LD}_{even}^d$  (resp.  $\mathcal{LD}_{odd}^d$ ) to be real vector spaces generated by isomorphism classes of link diagrams  $\Gamma$  of degree  $d$  modulo subspaces generated by the relations

- (1) If  $\Gamma$  contains more than one edge joining two vertices, then  $\Gamma = 0$ ;
- (2) If  $n$  is odd and  $\Gamma$  and  $\Gamma'$  are link diagrams such that a permutation of the vertices of  $\Gamma'$  results in a link diagram isomorphic to  $\Gamma$ , then

$$\Gamma = (-1)^\sigma \Gamma',$$

where

$$\begin{aligned} \sigma = & (\text{order of the permutation vertices}) \\ & + (\text{number of edges with different orientation}); \end{aligned}$$

- (3) If  $n$  is even and  $\Gamma$  and  $\Gamma'$  are link diagrams such that a permutation of the vertex and edge sets of  $\Gamma'$  result in a link diagram isomorphic to  $\Gamma$ , then

$$\Gamma = (-1)^\sigma \Gamma',$$

where

$$\begin{aligned} \sigma = & (\text{order of the permutation of segment vertices}) \\ & + (\text{order of the permutation of the edges}). \end{aligned}$$

Finally define the graded vector spaces

$$\mathcal{LD}_{even} = \bigoplus_d \mathcal{LD}_{even}^d \quad \text{and} \quad \mathcal{LD}_{odd} = \bigoplus_d \mathcal{LD}_{odd}^d.$$

When there is no danger of confusion, i.e. when  $n$  is understood, we will refer to both  $\mathcal{LD}_{even}^d$  and  $\mathcal{LD}_{odd}^d$  as  $\mathcal{LD}^d$  and to both  $\mathcal{LD}_{even}$  and  $\mathcal{LD}_{odd}$  as  $\mathcal{LD}$ .

*Remark 3.11.* The reader might argue that we should simply disallow multiple edges between a given pair of vertices rather than mod out by the subspace of such diagrams, but the differential, defined below, can introduce such edges.

*Remark 3.12.* Even in a fixed degree and after all the relations are imposed,  $\mathcal{LD}^d$  is still an infinite dimensional vector space: Consider for example the diagram consisting of three segments with one segment vertex on each segment, and a single free vertex with three edges which join it to the segment vertices. This is a diagram of degree zero. Overlaying copies of this diagram (that is, introducing new segment and free vertices and edges in a similar fashion) gives an infinite list of degree zero diagrams which are clearly independent in the vector space structure.

**3.2. Algebraic structures on the diagram complex.** We now discuss the differential and the product on the space of diagrams which will make it into a differential algebra.

**3.2.1. The differential.** The differential of a diagram will be a signed sum of diagrams obtained from the original by “contracting” certain edges or arcs. We begin with some terminology and conventions.

**Definition 3.13.** Let  $S$  be a nonempty set, and let  $s, t \in S$ . Define

$$R_{t \rightarrow s} : SP_2(S) \longrightarrow SP_2(S)$$

by

$$R_{t \rightarrow s}(T) = \begin{cases} T, & \text{if } t \notin T; \\ (T - \{t\}) \cup \{s\}, & \text{if } t \in T. \end{cases}$$

Thus the map  $R_{t \rightarrow s}$  replaces  $t$  with  $s$ . Let  $\Gamma = (V, E, b)$  be a link diagram and  $e$  be a mixed or free edge of  $\Gamma$ , or one of its arcs, and suppose  $b(e) = \{v, w\}$ , where  $v < w$  in the ordering of the vertices. In case  $e$  is an arc, we suppose it is represented by the pair  $(v, w)$ . Note that  $e$  necessarily joins distinct vertices.

**Definition 3.14.** With  $\Gamma$  and  $e$  as above, define  $\Gamma/e = (V', E', b')$  to be the link diagram such that

- $V' = V - \{w\}$  with the induced ordering of vertices,
- $E' = E - \{e\}$  with the induced ordering/orientation of edges (if applicable), and
- $b' = R_{w \rightarrow v} \circ b$ , restricted to  $E'$ .

We often refer to  $\Gamma/e$  as the diagram  $\Gamma$  with the edge/arc  $e$  contracted. The function  $R_{w \rightarrow v}$  above simply replaces an edge joining  $w$  with a vertex  $u$  with the edge which joins  $v$  to  $u$  instead. This can create a loop in the case of a chord between adjacent segment vertices when the arc between them is contracted. Note that the degree is increased by contraction of a mixed/free edge or arc: if  $\Gamma$  has degree  $d$ , then  $\Gamma/e$  has degree  $d + 1$ . The differential is a signed sum of diagrams made from  $\Gamma$  by contracting all possible edges and arcs. We will use the “position” function to help keep track of these signs.

**Definition 3.15.** Suppose  $S$  is a finite ordered set. Define the *position function* to be the unique order-preserving bijection

$$\text{pos} : S \rightarrow \{1, 2, \dots, |S|\}.$$

When  $x \in S$ , we write  $\text{pos}(x)$  for the value of this function at  $x \in S$ , or  $\text{pos}(x : S)$  when we wish to emphasize the underlying ordered set  $S$ .

**Definition 3.16.** The differential

$$(5) \quad \delta : \mathcal{LD}^d \longrightarrow \mathcal{LD}^{d+1}$$

is the unique linear extension to  $\mathcal{LD}^d$  of the map defined on a diagram  $\Gamma$  by

$$(6) \quad \delta(\Gamma) = \sum_{\text{free edges, mixed edges, and arcs } e \text{ of } \Gamma} \epsilon(e)\Gamma/e.$$

The number  $\epsilon(e)$  is equal to  $\pm 1$  depending on the parity of  $n$  and on the orderings of vertices and edges in the following way: Suppose the free/mixed edge or arc  $e$  connects vertices  $v$  and  $w$ .

- If  $n$  is odd and  $e$  is an edge or an arc oriented so the edge joins  $v$  to  $w$ , then

$$(7) \quad \epsilon(e) = \begin{cases} (-1)^{\text{pos}(w:V)}, & v < w, \\ -(-1)^{\text{pos}(v:V)}, & w < v. \end{cases}$$

- If  $n$  is even and  $e$  is a free edge or a mixed edge, then

$$(8) \quad \epsilon(e) = (-1)^{\text{pos}(e:E) + |V_{\text{free}}| + 1},$$

and if  $e$  is an arc, then

$$(9) \quad \epsilon(e) = (-1)^{\text{pos}(\max\{v,w\})}.$$

An example of the differential is given in Figure 2.

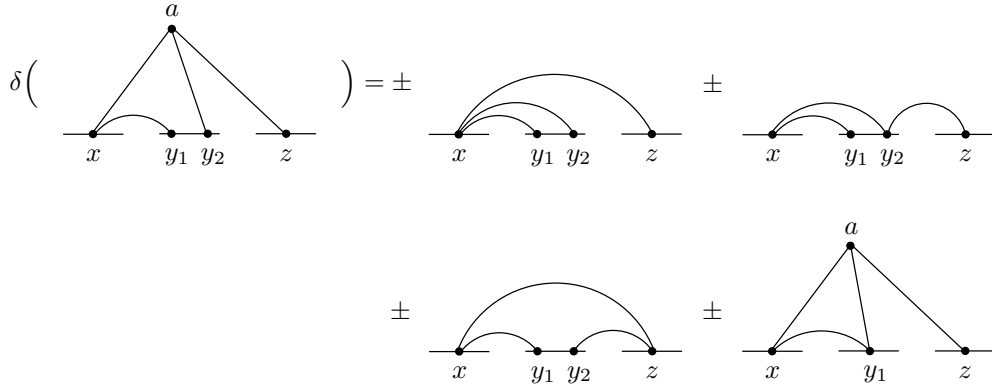


FIGURE 2. An example of the differential. The signs depend on the parity of  $n$ .

**3.2.2. The shuffle product.** The shuffle product on the space of diagrams associated to knots was first considered in [6]. Here we extend it to link diagrams as well as provide more details about its construction.

Consider two link diagrams  $\Gamma_1 = (V(\Gamma_1), E(\Gamma_1), b_{\Gamma_1})$  and  $\Gamma_2 = (V(\Gamma_2), E(\Gamma_2), b_{\Gamma_2})$ . Let

$$\text{seg}_i : V(\Gamma_i)_{\text{seg}} \longrightarrow \sqcup_i (I_i - \partial I_i)$$

be representatives of the equivalence class of the partition function for the segment vertices. Moreover, choose isomorphism class representatives for each diagram so that their vertex and edge sets are disjoint. Call an injective map

$$j : V(\Gamma_1)_{\text{seg}} \sqcup V(\Gamma_2)_{\text{seg}} \longrightarrow \sqcup_i (I_i - \partial I_i)$$

*admissible* if its restriction to  $V(\Gamma_i)_{\text{seg}}$  is in the same equivalence class as  $\text{seg}_i$  for  $i = 1, 2$ .

**Definition 3.17.** With  $\Gamma_1$  and  $\Gamma_2$  and an admissible map  $j$  as above, define

$$\Gamma_1 \cdot_j \Gamma_2 = (V(\Gamma_1 \cdot_j \Gamma_2), E(\Gamma_1 \cdot_j \Gamma_2), b_{\Gamma_1 \cdot_j \Gamma_2})$$

to be the diagram such that

- The set  $V(\Gamma_1 \cdot_j \Gamma_2) = V(\Gamma_1) \sqcup V(\Gamma_2)$ ;
- The set  $E(\Gamma_1 \cdot_j \Gamma_2) = E(\Gamma_1) \sqcup E(\Gamma_2)$ , and the orientations (if applicable) for edges are those induced by the orientations of elements of  $E(\Gamma_1)$  and  $E(\Gamma_2)$ ;
- The map  $b_{\Gamma_1 \cdot_j \Gamma_2} = b_{\Gamma_1} \sqcup b_{\Gamma_2}$ ;
- The set  $V(\Gamma_1 \cdot_j \Gamma_2)$  is decomposed as

$$V(\Gamma_1 \cdot_j \Gamma_2)_{seg} \sqcup V(\Gamma_1 \cdot_j \Gamma_2)_{free}$$

where

- $V(\Gamma_1 \cdot_j \Gamma_2)_{seg} = V(\Gamma_1)_{seg} \sqcup V(\Gamma_2)_{seg}$ , with ordering induced by the injection  $j$ ,
- $V(\Gamma_1 \cdot_j \Gamma_2)_{free} = V(\Gamma_1)_{free} \sqcup V(\Gamma_2)_{free}$ , with ordering induced by the ordered pair  $(V(\Gamma_1)_{free}, V(\Gamma_2)_{free})$ , and hence
- $V(\Gamma_1 \cdot_j \Gamma_2)$  is ordered by the ordered pair of ordered sets  $(V(\Gamma_1 \cdot_j \Gamma_2)_{seg}, V(\Gamma_1 \cdot_j \Gamma_2)_{free})$ ;
- The ordering of  $E(\Gamma_1 \cdot_j \Gamma_2)$  is that induced by the ordered pair  $(E(\Gamma_1), E(\Gamma_2))$  of ordered sets.

**Definition 3.18.** For link diagrams  $\Gamma_1$  and  $\Gamma_2$ , define their *shuffle product*  $\Gamma_1 \bullet \Gamma_2$  by

$$(10) \quad \Gamma_1 \bullet \Gamma_2 = \sum_{\text{admissible } j} \epsilon(\Gamma_1, \Gamma_2) \Gamma_1 \cdot_j \Gamma_2$$

where

$$\epsilon(\Gamma_1, \Gamma_2) = \begin{cases} (-1)^{|E(\Gamma_1)| |V(\Gamma_2)_{seg}|}, & n \text{ even}; \\ 1, & n \text{ odd}. \end{cases}$$

The proofs of the following two propositions are straightforward unravellings of the definitions.

**Proposition 3.19.** *The shuffle product is skew-commutative; that is,*

$$\Gamma_1 \bullet \Gamma_2 = (-1)^{|\Gamma_1| |\Gamma_2|} \Gamma_2 \bullet \Gamma_1$$

where

$$|\Gamma| = \begin{cases} |E(\Gamma)| + |V(\Gamma)_{seg}|, & n \text{ even}; \\ |V(\Gamma)_{free}| + |V(\Gamma)_{seg}|, & n \text{ odd}. \end{cases}$$

**Proposition 3.20.** *The differential  $\delta$  is a derivation. That is,*

$$\delta(\Gamma_1 \bullet \Gamma_2) = \delta(\Gamma_1) \bullet \Gamma_2 + (-1)^{|\Gamma_1|} \Gamma_1 \bullet \delta(\Gamma_2).$$

Hence

**Proposition 3.21.** *The diagram complex  $(\mathcal{LD}, \delta, \bullet)$  is a commutative differential graded algebra (CDGA) with unit and its cohomology  $H^*(\mathcal{LD})$  is thus a commutative graded algebra.*

*Remark 3.22.* There is also a coproduct on  $\mathcal{LD}$ , analogous to the one given in [6]. Since we will not use this structure (shuffle product, on the other hand, will be needed in [20]), we will only remark that this should give  $\mathcal{LD}$  the structure of a Hopf algebra, and the map appearing in Theorem 4.28 becomes a map of Hopf algebras.

**3.3. A subcomplex for the space of homotopy string links.** A homotopy string link need not be an embedding. As such, integration over  $\mathcal{H}_m^n$  will not be possible in as general a way as prescribed on the complex  $\mathcal{LD}$  (see Section 4.2 for more details) due to possible self-intersections of each component of the link. In this section we will identify a subcomplex  $\mathcal{HD}$  of  $\mathcal{LD}$  for which it will be possible to carry out the integration and construct elements of  $\Omega^*(\mathcal{H}_m^n)$ .

**Definition 3.23.** Define the space of *homotopy link diagrams*, denoted  $\mathcal{HD}$ , to be the subspace of  $\mathcal{LD}$  generated by diagrams  $\Gamma$  which

- (1) contain no loops; and
- (2) satisfy the condition that if there exists a path between distinct vertices on a given segment, then it must pass through a vertex on another segment.

Some examples of homotopy link diagrams are given in Figure 3.

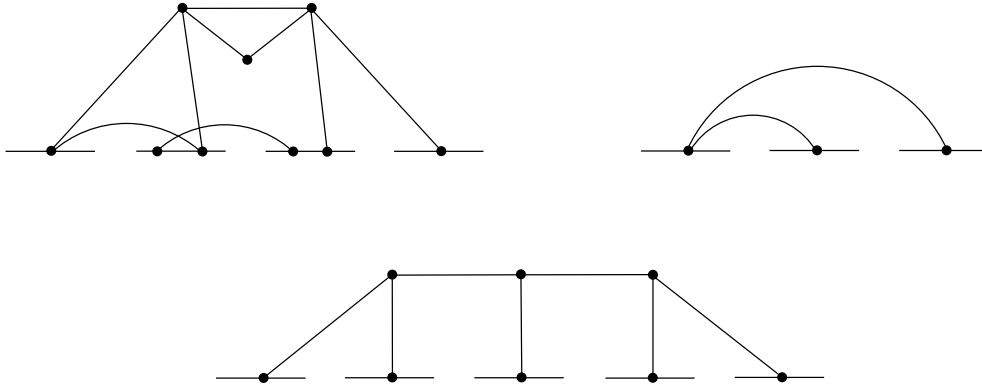


FIGURE 3. Some examples of homotopy link diagrams (without decorations). The bottom one is a tree of the sort that will give rise to finite type invariants in Section 5.

**Proposition 3.24.**  $\mathcal{HD}$  is a differential subalgebra of  $\mathcal{LD}$ .

*Proof.* To show  $\mathcal{HD}$  is a subcomplex of  $\mathcal{LD}$ , we must show that  $\delta(\mathcal{HD}) \subset \mathcal{HD}$ . Write  $\Gamma = (V, E, b)$  and  $\Gamma/e = (V', E', b')$ , where  $e = \{v, w\}$  with  $v < w$ . Suppose, on the contrary, that  $\Gamma/e$  is not an element of  $\mathcal{HD}$ . There are two cases. If  $\Gamma/e$  has a loop, then if it is at a free vertex,  $\Gamma$  itself must either have a loop or  $\Gamma$  must have multiple edges between a pair of free vertices. The first case is impossible and in the second case  $\Gamma$  is set to zero.

The second case is where  $v_1, v_2$  are distinct segment vertices lying on the same segment of  $\Gamma/e$ , and there is a path  $\alpha = \{e_i\}_{i=1}^k$  of edges from  $v_1$  to  $v_2$  which does not pass through a vertex on a different segment than the one on which  $v_1$  and  $v_2$  lie. In this case it is enough to show

that there is a path between  $v_1$  and  $v_2$  in  $\Gamma$  which also does not pass through a vertex lying on a different segment.

Let  $\alpha = \{e_i\}_{i=1}^k$  be a path from  $v_1$  to  $v_2$  in  $\Gamma/e$ , so that  $v_1 \in b'(e_1), v_2 \in b'(e_k)$  and  $b'(e_i) \cap b'(e_{i+1}) \neq \emptyset$  for all  $i$ . If  $\alpha$  also has the property that  $v_1 \in b(e_1), v_2 \in b(e_k)$  and  $b(e_i) \cap b(e_{i+1}) = \emptyset$  for all  $i$ , then we have a path between  $v_1$  and  $v_2$  in  $\Gamma$ , contradicting the fact that  $\Gamma \in \mathcal{HD}$ . Otherwise,  $\alpha$  is a sequence of edges in  $\Gamma$  such that  $v_1 \in b(e_1)$  and  $v_2 \in b(e_k)$ . Let  $j$  be the smallest integer such that  $b(e_j) \cap b(e_{j+1}) = \emptyset$ . Since  $b'(e_j) \cap b'(e_{j+1}) \neq \emptyset$ , and since  $b' = R_{w \rightarrow v} \circ b$ , we must have that  $w \in b(e_j)$  or  $w \in b(e_{j+1})$ . Without loss of generality assume  $w \in b(e_j)$ . Then it must be that  $v \in b(e_{j+1})$ , and in this case the edge  $e$  has the property that  $b(e_j) \cap b(e) \neq \emptyset$  and  $b(e_{j+1}) \cap b(e) \neq \emptyset$ , so that  $\{e_1, \dots, e_j, e, e_{j+1}\}$  forms a path in  $\Gamma$ . Continuing in this manner, we can construct a path  $\alpha'$  between  $v_1$  and  $v_2$  in  $\Gamma$ . Note that the only new vertices which  $\alpha'$  passes through are the vertex  $w$  itself. We will be done if we can argue that  $w$  cannot be a segment vertex lying on a segment different from  $v_1$  and  $v_2$  unless the path  $\alpha$  already passed through such a segment vertex. But this is clear: if  $w$  is such a vertex, then since  $v < w$ , we must have that  $v$  is also a vertex lying on the same segment (because  $e$  is necessarily the arc between them), in which case the original path passes through  $v$ .

That  $\mathcal{HD}$  is closed under the shuffle product is clear since this product does not create new paths of edges.  $\square$

A few words of clarification and justification for Definition 3.23 are in order. Our definition of  $\mathcal{HD}$  excludes diagrams which contain a chord connecting two vertices on a single segment. It also excludes all possible diagrams which, via contractions of edges, might produce such a chord. What we are trying to capture geometrically are linking phenomena which “ignore” the knotting of each strand. The reason for this is simple: there is no knotting of individual strands in  $\mathcal{H}_m^n$ , as they may pass through themselves. Once integration over diagrams is defined in Section 4.4, it will be clear that a chord between segment vertices captures something about linking between those segments. So when the segment vertices lie on the same segment, this means a chord between them captures something about knotting, or self-linking, of that segment. Similarly, integrals that correspond to loops will also only contain information about single strands.

**3.4. Diagram complexes in degree zero.** In Section 5 we will focus on the case  $n = 3$  of classical links and see what link invariants, namely elements of  $H^0(\mathcal{L}_m^3)$  and  $H^0(\mathcal{H}_m^3)$  we can obtain via configuration space integrals from our diagram complexes. As will see in Section 4.4, when  $n = 3$ , degree zero diagrams will correspond with degree zero forms (although the degree will in general not be preserved), so we want

$$(11) \quad 0 = 2|E(\Gamma)| - 3|V(\Gamma)_{free}| - |V(\Gamma)_{seg}|.$$

It was already noted in the discussion following equation (4) in Section 3.1 that these are precisely the trivalent diagrams.

To see what is in the kernel of the differential in degree zero, we first introduce another grading on  $\mathcal{LD}$  and  $\mathcal{HD}$  in order to restrict our attention to certain finite-dimensional subspaces of these spaces.

**Definition 3.25.** Define *order* of a diagram  $\Gamma$  to be

$$\text{ord}(\Gamma) = |E(\Gamma)| - |V(\Gamma)_{free}|.$$



It is easy to see if  $\Gamma$  has order  $k$ , then each summand of  $\delta(\Gamma)$  has order  $k$  as well. Thus, for each  $k = \text{ord}(\Gamma)$ , we get subcomplexes  $\mathcal{LD}_k$  and  $\mathcal{HD}_k$  of  $\mathcal{LD}$  and  $\mathcal{HD}$  respectively.

Note that, in the case  $n = 3$  of interest here, and in degree zero, we also necessarily have

$$\text{ord}(\Gamma) = \frac{1}{2}(|V(\Gamma)_{\text{seg}}| + |V(\Gamma)_{\text{free}}|).$$

This means that  $\mathcal{LD}_k^0$  and  $\mathcal{HD}_k^0$  are finite-dimensional since the above equation fixes the number of vertices, and edges cannot be added indefinitely since a diagram with more than one edge joining a pair of vertices is set to zero.

Thus to understand the kernel of the differential  $\delta$  in degree zero, it suffices to understand the cokernel of its adjoint in each degree. It is not hard to see that this adjoint “blows up” four-valent vertices in all possible ways.

In the case of  $\mathcal{LD}_k^0$ , the possibilities are given in Figures 4, 5, and 6. The signs of the resulting diagrams arise from the labeling conventions associated to edge contractions (in particular recall that free vertices always have higher label than segment ones, so  $i < j$  in the left picture on the bottom of of Figure 4). (In the figures, the triples of diagrams resulting from the blowup of a vertex are the same outside of the pictured portions.)

Therefore  $\mathcal{LD}_k^0$  is for each  $k$  generated by trivalent diagrams satisfying the relations that the sum of the three diagrams in Figures 4 and 5 is zero and that the diagram on the bottom of Figure 6 is zero. As is customary, we will call those *STU*, *IHX*, and *1T* relations, respectively.

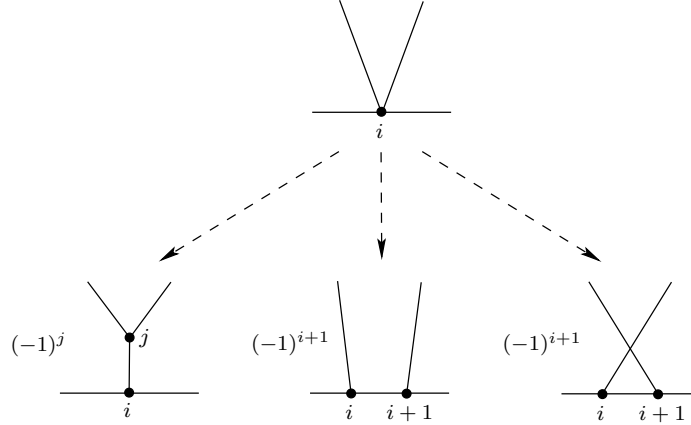


FIGURE 4. Blowups giving rise to the STU relation.

*Remark 3.26.* Bar-Natan [2] has shown that the IHX and 1T relations follows from the STU relation.

We now consider the case of  $\mathcal{HD}_k^0$ , where there are some additional observations to be made. First, the 1T relation is now vacuous here since  $\mathcal{HD}$  contains no diagrams with chords connecting vertices on the same segment. Second, suppose that the two loose edges in the top diagram of Figure 4 belong to a loop of edges with all vertices except  $i$  free. This is depicted in Figure 7.

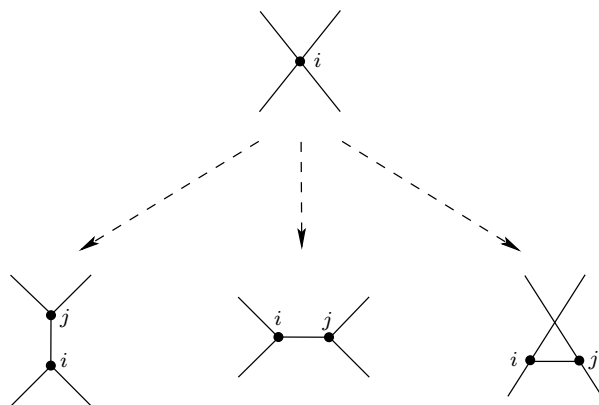


FIGURE 5. Blowups giving rise to the IHX relation.

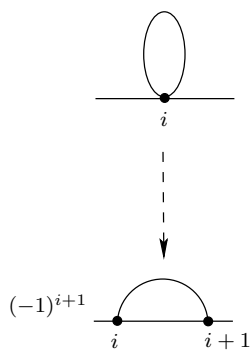


FIGURE 6. Blowup giving rise to the 1T relation.

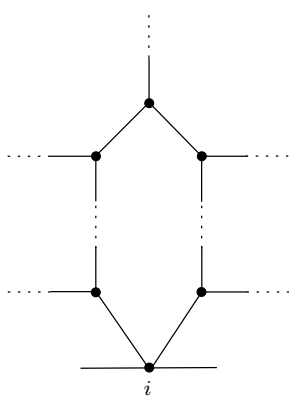


FIGURE 7.

Then blowing up vertex  $i$  can only result in one diagram, namely the leftmost one from the STU relation. The other two would be diagrams with paths between two segment vertices on

the same segment that only go through free vertices, and such diagrams are not elements of  $\mathcal{HD}$ . We thus get the special case of the STU relation in  $\mathcal{HD}$  given in Figure 8.

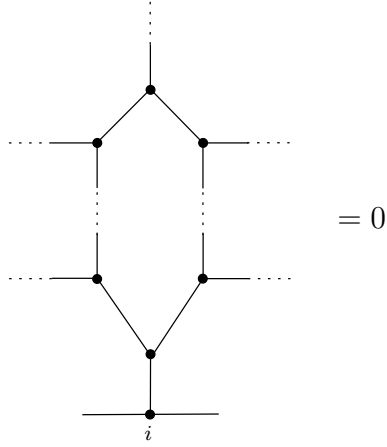


FIGURE 8.

This relation extends to all diagrams with loops of free edges and not just those that are separated from a segment by a single mixed edge. Namely, the STU relation can be applied repeatedly to any path between the loop of free edges and a segment (there are always such paths since every free vertex must have a path to a segment vertex) and the situation can be reduced to that of Figure 8. An example is given in Figure 9.

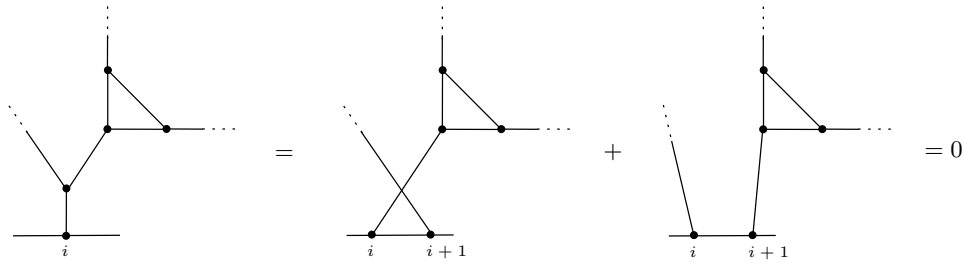


FIGURE 9. An example of the vanishing of a diagram with a loop of free edges.

*Remark 3.27.* At first glance, it might seem that the diagram from Figure 7 should not be permitted in  $\mathcal{HD}$  since repeated contractions of its edges would eventually produce a loop at vertex  $i$ , and loops have been excluded from  $\mathcal{HD}$ . However, such contractions would first produce a double edge between vertex  $i$  and another free vertex, and a diagram with a double edge would already be zero by definition of  $\mathcal{LD}$ .

*Remark 3.28.* There is another interesting consequence of the STU relation in  $\mathcal{HD}_k^0$  which we will have use for in [20] but not the present work. Namely, suppose that the same two loose edges in the top diagram of Figure 4 end on the same segment. In other words, suppose the

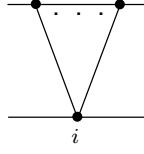
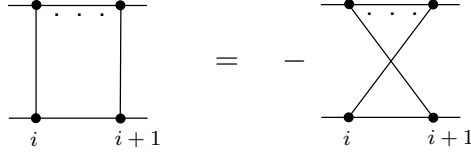


FIGURE 10.

picture is as in Figure 10, where the dotted lines indicate that there might be other segment vertices between those pictured.

Then the only two possible blowups are the middle and the right diagrams in Figure 4 since the left contains a path between two segment vertices that goes only through a free vertex. We thus get a special case of the STU relation in  $\mathcal{HD}_k^0$ , given in Figure 11.


 FIGURE 11. A consequence of the STU relation in  $H^0(\mathcal{HD})$ .

We can now collect the observations made so far into the following

**Proposition 3.29.** *For each  $k \geq 0$ ,*

- $\mathcal{LD}_k^0$  is generated by trivalent diagrams  $\Gamma$  modulo the
  - STU relation;
  - IHX relation;
  - 1T relation.
- $\mathcal{HD}_k^0$  is generated by trivalent diagrams  $\Gamma$  modulo the
  - STU relation;
  - IHX relation;
  - If  $\Gamma$  contains a closed path of edges, then  $\Gamma = 0$ ;

*Remarks 3.30.*

- (1) As mentioned in the Introduction, these descriptions of  $\mathcal{LD}_k^0$  and  $\mathcal{HD}_k^0$  already exist in the literature [3, 14, 16, 27], although Mellor [14] and Mellor-Thurston [16] work with the variant of  $\mathcal{HD}_k^0$  consisting of trivalent diagrams without segments and without the STU relation (but they keep the other relations). In fact, the only reason we listed the last relation for  $\mathcal{HD}_k^0$  (we could have left it out since it follows from the STU relation) is so that our description would exactly match those in [14, 16].
- (2) Define a *tree* to be a diagram such that every path of edges joining any two vertices is unique, and define a *leaf* to be a mixed edge or chord of a tree (so a leaf has at least one associated segment vertex). Define a *forest* to be a diagram whose connected

components are all trees. Since elements of  $\mathcal{HD}_k^0$  are (sums of) trivalent diagrams without loops, every element is a sum of forests, each of whose trees has at most  $m$  leaves, where  $m$  is the number of distinct segments, and such that the segment vertices associated with the leaves all lie in distinct segments (that is, there is at most one segment vertex on each segment for a given tree in the forest). This was alluded to in the description of Figure 3, where the bottom diagram is such a tree.

In order to obtain real-valued (rather than diagram-valued) invariants, we will be considering functionals on  $\mathcal{LD}_k^0$  and  $\mathcal{HD}_k^0$ .

**Definition 3.31.** Define the spaces of *degree  $k$  link weight systems*  $\mathcal{LW}_k$  and *degree  $k$  homotopy link weight systems*  $\mathcal{HW}_k$  as

$$\mathcal{LW}_k = (\mathcal{LD}_k^0)^* \quad \text{and} \quad \mathcal{HW}_k = (\mathcal{HD}_k^0)^*$$

respectively. Here  $(-)^*$  denotes the dual vector space.

Thus  $\mathcal{LW}_k$  and  $\mathcal{HW}_k$  consist of the functionals on  $\mathcal{LD}_k^0$  and  $\mathcal{HD}_k^0$ , respectively, that vanish on the relations from Proposition 3.29.

*Remark 3.32.* The real reason we introduced the grading by order is that weight systems of order  $k$  are precisely finite type  $k$  invariants; see Theorems 5.6 and 5.8.

One last observation we need to make is that, instead of considering trivalent diagrams, one can always reduce to the case of diagrams containing only chords, i.e. *chord diagrams*. That is, note that any trivalent diagram can be rewritten as a sum of chord diagrams using the STU relation. The resulting complex inherits a different relation as follows: Because the trivalent diagram in the STU relation can have both of its “loose” edges also ending in segments (necessarily different segments in the case of  $\mathcal{HD}$ ), applying the STU relation twice gives what is known as the  $4T$  relation, depicted in Figure 12.

Denote by

$$\mathcal{LC}_k^0 \quad \text{and} \quad \mathcal{HC}_k^0$$

the  $\mathbb{R}$ -vector spaces generated by chord diagrams on  $m$  segments with  $k$  chords ending on  $2k$  distinct vertices (since degree zero implies trivalence, two chords cannot end in a common segment vertex). For the latter space, there can be no chords with both endpoints on the same segment. We will call these the *link chord diagrams* and *homotopy link chord diagrams*. Using the relationship between the STU and  $4T$  relations, we have the following straightforward generalization of [2, Theorem 6]:

**Theorem 3.33.** *There are isomorphisms*

$$\mathcal{LD}_k^0 \cong \mathcal{LC}_k^0 / (4T, 1T) \quad \text{and} \quad \mathcal{HD}_k^0 \cong \mathcal{HC}_k^0 / 4T$$

*given by sending a diagram with no free vertices (a chord diagram) to itself and a diagram with free vertices to the sum of chord diagrams obtained from it via the STU relation.*

Now denote by

$$\mathcal{LCW}_k \quad \text{and} \quad \mathcal{HCW}_k$$

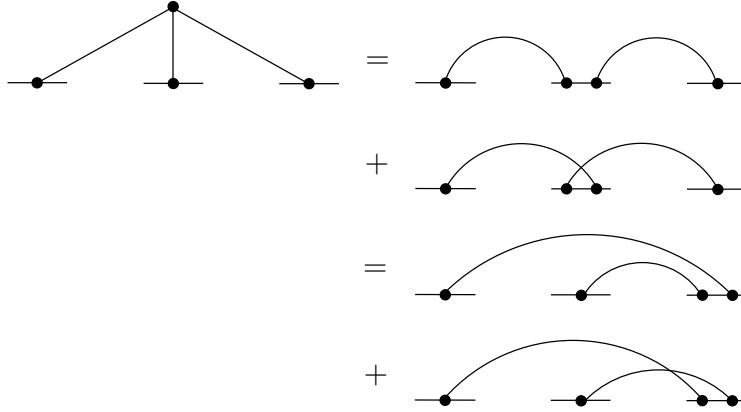


FIGURE 12. Applying the STU relation to the middle and the right mixed edge produces the equality of the two pairs of chord diagrams. Any time four chord diagrams differ in two places as pictured, one obtains such an equality, called the 4T relation. The three arcs belong to distinct segments in the case of  $\mathcal{HD}$  and some or all of them could belong to same segment in the case of  $\mathcal{LD}$ .

the weight systems (linear functionals) on  $\mathcal{LC}_k^0/(4T, 1T)$  and  $\mathcal{HC}_k^0/4T$ , respectively. Dualizing Theorem 3.33, we thus have isomorphisms

$$(12) \quad \mathcal{LW}_k \cong \mathcal{LCW}_k \quad \text{and} \quad \mathcal{HW}_k \cong \mathcal{HCW}_k.$$

Theorem 3.33 will be used in the proof of Theorem 5.8.

#### 4. CONFIGURATION SPACE INTEGRALS AND COHOMOLOGY OF HOMOTOPY STRING LINKS

**4.1. Compactification of configuration spaces.** In this section we review the standard construction of a compactification of configuration spaces over which we will integrate to produce invariants. This is necessary since integrals over the ordinary open configuration space may not converge. The original compactification is due to Fulton and MacPherson [7, 1], but we follow Sinha's [24] alternative construction (also considered by Kontsevich and Soibelman [12]).

**Definition 4.1.** Let

$$C(p, \mathbb{R}^n) = \{(x_1, x_2, \dots, x_p) \in (\mathbb{R}^n)^p : x_i \neq x_j \text{ for } i \neq j\},$$

be the configuration space of  $p$  points in  $\mathbb{R}^n$ . When  $n = 1$ , the configuration space has  $p!$  components, and in this case  $C(p, \mathbb{R})$  will mean the component consisting of those  $(x_1, \dots, x_p)$  such that  $x_1 < \dots < x_p$ .

For all  $1 \leq i < j < k \leq p$  we have maps

$$(13) \quad v_{ij} = \frac{x_j - x_i}{|x_j - x_i|} \in S^{n-1}, \quad a_{ijk} = \frac{|x_i - x_j|}{|x_i - x_k|} \in [0, \infty],$$

whose domain is  $C(p, \mathbb{R}^n)$  and where  $[0, \infty]$  denotes the one-point compactification of  $[0, \infty)$ . These maps will measure the direction and relative rates of collision of configuration points

respectively. Adding this information to the configuration space is achieved by considering the map

$$(14) \quad \gamma: C(p, \mathbb{R}^n) \longrightarrow (\mathbb{R}^n)^p \times (S^{n-1})^{(p)}_{(2)} \times [0, \infty]^{(p)}_{(3)} \\ (x_1, \dots, x_p) \longmapsto (x_1, \dots, x_p, v_{12}, \dots, v_{ij}, \dots, v_{(p-1)p}, a_{123}, \dots, a_{ijk}, \dots, a_{(p-2)(p-1)p}).$$

**Definition 4.2.** Define  $C[p, \mathbb{R}^n]$  to be the closure of the image of  $\gamma$ . That is,

$$C[p, \mathbb{R}^n] = \overline{\gamma(C(p, \mathbb{R}^n))} \subset (\mathbb{R}^n)^p \times (S^{n-1})^{(p)}_{(2)} \times [0, \infty]^{(p)}_{(3)}.$$

Since  $\mathbb{R}^n$  is itself not compact, we think of  $C[p, \mathbb{R}^n]$  as the subspace of  $C[p+1, S^n]$  where  $S^n = \mathbb{R}^n \cup \infty$  and the last configuration point is the point at infinity. This is why we will have to consider the faces obtained by points escaping to infinity.

Here are some properties of  $C[p, \mathbb{R}^n]$  that are relevant for our purposes. Proofs can be found in [24]:

- (1) The space  $C[p, \mathbb{R}^n]$  is a manifold with corners homotopy equivalent to  $C(p, \mathbb{R}^n)$ ;
- (2) The boundary of  $C[p, \mathbb{R}^n]$  is given by points colliding or escaping to infinity;
- (3) The directions and relative rates of collision are recorded, so that a  $k$ -stage collision (points coming together or going to infinity in  $k$  different stages rather than all of them doing this at the same instance) gives a point in codimension  $k$  stratum of  $C[p, \mathbb{R}^n]$ ;

The last property in particular says that codimension one faces of  $C[p, \mathbb{R}^n]$  consist of configurations where some subset of the points has come together or escaped to infinity at the same time. These faces are of particular interest since they play a role in checking whether some differential form obtained on the space of links is closed (i.e. they are relevant for an application of Stokes' Theorem).

**4.2. Bundles of compactified configuration spaces.** Given  $\Gamma \in \mathcal{LD}$ , we will construct in this section a certain bundle of configuration spaces over  $\mathcal{L}_m^n$ . There is already a standard recipe for doing this which was initiated in the case of knots ( $m = 1$ ) in [4] and fully developed in [5]. A straightforward generalization to links ( $m > 1$ ) was then given in [27]. However, this fails to even produce a bundle over  $\mathcal{H}_m^n$  by restriction to the subcomplex  $\mathcal{HD}$  as we will see in Section 4.2.4.

To fix this issue, we devise a more refined way of constructing bundles which works over both  $\mathcal{L}_m^n$  and  $\mathcal{H}_m^n$ . The standard way of doing this, as in [5, 27], is our starting point, and is described in Section 4.2.1 (this material essentially comes from Section 3.2 of [27]). In Section 4.2.4 we refine this construction to produce bundles over spaces of homotopy string links. The difference between the two approaches can be summarized very succinctly: in the standard approach, only vertices of a diagram were taken into account in the construction of bundles, and in the new approach, we will take into account both vertices and edges. We will show the compatibility of the approaches in Section 4.4.

**4.2.1. Bundles of compactified configuration spaces from vertices of a diagram.** A diagram  $\Gamma \in \mathcal{LD}$  will define a space where the segment vertices of  $\Gamma$  correspond to configuration points moving along a link in  $\mathbb{R}^n$  and free vertices correspond to configuration points that are free to move anywhere in  $\mathbb{R}^n$ .



Suppose  $\Gamma \in \mathcal{LD}$  has  $i_j$  segment vertices on the  $j$ th segment,  $1 \leq j \leq m$ , and  $s$  free vertices. The evaluation map

$$(15) \quad ev_\Gamma: \mathcal{L}_m^n \times \prod_{j=1}^m C[i_j, \mathbb{R}] \longrightarrow C \left[ \sum_{j=1}^m i_j, \mathbb{R}^n \right]$$

is given on the interiors of  $C[i_k, \mathbb{R}^n]$  by evaluating the  $k$ th strand of a link  $L \in \mathcal{L}_m^n$  on  $i_k$  configuration points, and then extending to the boundary of each  $C[i_k, \mathbb{R}^n]$  by continuity. That is, it is the extension of the map on the open configuration space given by

$$(L, (x_1^1, \dots, x_{i_1}^1), \dots, (x_1^m, \dots, x_{i_m}^m)) \mapsto (L(x_1^1), \dots, L(x_{i_1}^1), \dots, L(x_1^m), \dots, L(x_{i_m}^m)).$$

We also have the projection

$$(16) \quad pr: C \left[ \sum_{j=1}^m i_j + s, \mathbb{R}^n \right] \longrightarrow C \left[ \sum_{j=1}^m i_j, \mathbb{R}^n \right]$$

given by forgetting the last  $s$  points of a configuration, as well as all the  $v_{ij}$  and  $a_{ijk}$  which involve any of the last  $s$  points.

**Definition 4.3.** Given  $\Gamma \in \mathcal{LD}$  with  $i_j$  segment vertices on the  $j$ th segment and  $s$  free vertices, let  $\vec{i} = (i_1, \dots, i_m)$ , and let

$$C[\vec{i} + s; \mathcal{L}_m^n, \Gamma]$$

be the pullback

$$(17) \quad \begin{array}{ccc} C[\vec{i} + s; \mathcal{L}_m^n, \Gamma] & \longrightarrow & C \left[ \sum_{j=1}^m i_j + s, \mathbb{R}^n \right] \\ \downarrow & & \downarrow pr \\ \mathcal{L}_m^n \times \prod_{j=1}^m C[i_j, \mathbb{R}] & \xrightarrow{ev_\Gamma} & C \left[ \sum_{j=1}^m i_j, \mathbb{R}^n \right] \end{array}$$

We then have the following special case of Proposition A.3 in [4].

**Proposition 4.4.** *With  $\Gamma$  as above, the projection*

$$\pi_{\mathcal{L}, \Gamma}: C[\vec{i} + s; \mathcal{L}_m^n, \Gamma] \longrightarrow \mathcal{L}_m^n$$

*is a fiber bundle with fiber a finite-dimensional smooth manifold with corners.*

We will denote the fiber of  $\pi_{\mathcal{L}, \Gamma}$  over a link  $L$  by

$$\pi_{\mathcal{L}, \Gamma}^{-1}(L) = C[\vec{i} + s; L, \Gamma].$$

We think of this space as a configuration space whose first  $i_1$  points must lie on the first strand of  $L$ , second  $i_2$  must lie on the second strand, and so on, while the last  $s$  are free to move anywhere in  $\mathbb{R}^n$  (including on the image of  $L$ ).

4.2.2. *Bundles from diagram vertices and a difficulty with homotopy links.* If  $\Gamma$  is a diagram in  $\mathcal{HD}$ , then the above construction will not in general produce a fiber bundle over  $\mathcal{H}_m^n$ . The first problem is that a generic element  $H \in \mathcal{H}_m^n$  need not be an embedding or even an immersion, so that the target of the evaluation map is not the usual compactified configuration space, but rather a “partial” configuration space where some points are allowed to collide (without regard for how), while others are not. The second problem, not as easily overcome, is that the map from one partial configuration space to another which restricts to some subset of the original set of points is usually not a fibration, making it difficult to produce a fiber bundle by pullback. As an illustration, consider the following example.

**Example 4.5.** Define

$$C(2, 1; \mathbb{R}^n) = \{(x_1, x_2, y) \in (\mathbb{R}^n)^3 : x_1, x_2 \neq y\}.$$

and let  $C[2, 1; \mathbb{R}^n]$  denote its compactification (we only compactify along the diagonals which have been removed). Next, take  $m = 1$  (so there is one strand) and any value of  $n$ , and consider the evaluation map  $ev : \mathcal{H}_1^n \times C[2, \mathbb{R}] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ . The projection  $pr : C[2, 1; \mathbb{R}^n] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  to the first two coordinates is not even a fibration, as the fiber over a point  $(x_1, x_2)$  with  $x_1 = x_2$  is homotopy equivalent to  $S^{n-1}$ , while the fiber over such a pair with  $x_1 \neq x_2$  is homotopy equivalent to  $S^{n-1} \vee S^{n-1}$ . The problem persists with links of more components.

However, if we only allow one point on each strand for the evaluation map, then we can proceed as follows. We have an evaluation map

$$ev : \mathcal{H}_m^n \times \prod_{j=1}^m C[1, \mathbb{R}] \longrightarrow C[m, \mathbb{R}^n]$$

obtained by evaluating each strand of a homotopy link on exactly one point in that strand. The image necessarily lies in the interior of the compactified configuration space  $C[m, \mathbb{R}^n]$  since the images of the  $m$  strands are disjoint.

We again have a projection map

$$(18) \quad pr : C[m + s, \mathbb{R}^n] \longrightarrow C[m, \mathbb{R}^n]$$

which is a fibration (of manifolds with corners) so that one can form the pullback

$$\begin{array}{ccc} C[\vec{1} + s; \mathcal{H}_m^n] & \longrightarrow & C[m + s, \mathbb{R}^n] \\ \downarrow & & \downarrow pr \\ \mathcal{H}_m^n \times \mathbb{R}^m & \xrightarrow{ev} & C[m, \mathbb{R}^n] \end{array}$$

There is now a bundle

$$(19) \quad C[\vec{1} + s; \mathcal{H}_m^n] \longrightarrow \mathcal{H}_m^n$$

for the same reason we have one in Proposition 4.4 (it should be noted that A3 of [4] may appear to the reader not to apply, but it depends on A5, which does apply in this situation and gives the result we claim). We now use this observation to build bundles over  $\mathcal{H}_m^n$  for any diagram  $\Gamma \in \mathcal{HD}$ , and this will naturally extend to diagrams in  $\mathcal{LD}$ . In order to do so, we need to break our diagrams up into pieces, called “grafts”.

#### 4.2.3. The graft components of a diagram.

**Definition 4.6.** For a vertex  $v$  in a diagram  $\Gamma$ , let  $N(v)$  be the set of all pairs  $(w, e)$  such that  $b(e) = \{v, w\}$ .

Thus  $N(v)$  consists of all the “neighbors” of  $v$  counted with multiplicity according to edges.

**Definition 4.7.** Let  $\Gamma = (V, E, b) \in \mathcal{LD}$  be a diagram. Define the *hybrid* of  $\Gamma$  to be the diagram  $\tilde{\Gamma} = (\tilde{V}, \tilde{E}, \tilde{b})$  defined as follows: The set  $\tilde{V}$  is obtained from  $V$  by replacing each segment vertex  $v \in V$  of  $\Gamma$  with the set  $v \times N(v)$ , the elements of which will represent new vertices, if  $N(v) \neq \emptyset$ , and otherwise the vertex set is unchanged. The edge set  $\tilde{E}$  is equal to  $E$ . The map  $\tilde{b}$  is induced from  $b$  according to the following rule: Suppose  $b(e) = \{v, w\}$ . If  $v, w \in \tilde{V}$ , then  $\tilde{b}(e) = b(e)$ . If one of  $v$  or  $w$ , say  $v$ , is a segment vertex, then  $\tilde{b}(e) = \{(v, (w, e)), w\}$ . If both are, then  $\tilde{b}(e) = \{(v, (w, e)), (w, (v, e))\}$ .

The hybrid is not a link diagram, but it does induce certain link diagrams which are subdiagrams of the original link diagram  $\Gamma$ .

**Definition 4.8.** For a diagram  $\Gamma \in \mathcal{LD}$  with hybrid  $\tilde{\Gamma}$ , define the *graft components* of  $\tilde{\Gamma}$  to be the set of all subdiagrams of  $\tilde{\Gamma}$  determined by its path components. That is, the graft component containing the vertex  $v$  consists of all vertices of  $\tilde{\Gamma}$  which can be joined by a path in  $\tilde{\Gamma}$  to  $v$ , and all edges which can appear in such paths.

**Example 4.9.** Consider the diagram  $\Gamma$  in Figure 13. The five graft components of its hybrid  $\tilde{\Gamma}$  are given in Figure 14.

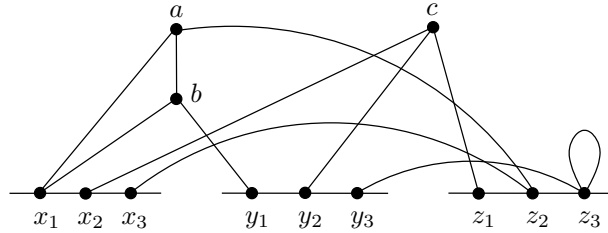


FIGURE 13.

The following is clear by construction.

**Proposition 4.10.** Each chord of  $\Gamma$  gives rise to a graft component consisting of two vertices and a single edge, and each loop at a segment vertex gives rise to a graft component with a single vertex and a single edge.

Although the hybrid  $\tilde{\Gamma}$  is not a link diagram, each graft component  $c(\tilde{\Gamma})$  of  $\tilde{\Gamma}$  canonically defines an element of  $\mathcal{LD}$ , with its structure induced by  $\Gamma$ .

**Definition 4.11.** Suppose the diagram  $c(\tilde{\Gamma}) = (V(c(\tilde{\Gamma})), E(c(\tilde{\Gamma})), b_{c(\tilde{\Gamma})})$  is a graft component of  $\tilde{\Gamma}$ , so that  $V(c(\tilde{\Gamma})) \subset \tilde{V}$  and  $E(c(\tilde{\Gamma})) \subset \tilde{E} = E$ . The forgetful map  $\tilde{V} \rightarrow V$  identifies  $c(\tilde{\Gamma})$  with a subdiagram  $c(\Gamma)$  of  $\Gamma$ , called a *graft* of  $\Gamma$  which inherits all the necessary structure for it to define an element of  $\mathcal{LD}$ .

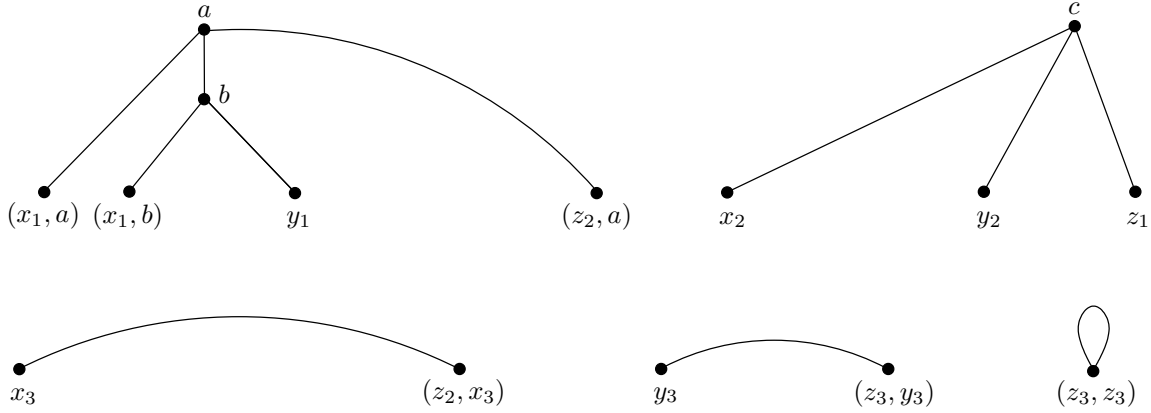


FIGURE 14. The five graft components of the diagram in 13. We have simplified the labels on the vertices of the graft components because the original diagram does not possess multiple edges between a given pair of vertices.

If  $\Gamma \in \mathcal{HD}$ , then it is clear that all the grafts of  $\Gamma$  are also elements of  $\mathcal{HD}$ . The set of all graft components, and hence the set of all grafts, can be ordered according to the ordering of the vertices of  $\Gamma$ ; no two grafts will have the same underlying vertex sets because diagrams with multiple edges between a pair of vertices are set to zero.

If  $\Gamma \in \mathcal{HD}$ , the grafts of  $\Gamma$  have an additional useful property which will allow us to build bundles over  $\mathcal{H}$ .

**Proposition 4.12.** *For  $\Gamma \in \mathcal{HD}$ , each graft of  $\Gamma$  has at most one segment vertex on each segment.*

*Proof.* First we claim that for any pair of distinct free vertices  $v, v'$  in the same graft component  $c(\tilde{\Gamma})$ , there exists a path of free edges between them. This is clear since each vertex of  $\tilde{\Gamma}$  which arises from a segment vertex of  $\Gamma$  is univalent, so any path between  $v$  and  $v'$  in  $c(\tilde{\Gamma})$  can be shortened to avoid such vertices. This clearly descends to a path in  $c_\Gamma$  between  $v$  and  $v'$  consisting only of free edges.

Now suppose, on the contrary, that there is some graft component  $c(\tilde{\Gamma})$  of  $\tilde{\Gamma}$  such that the associated graft  $c(\Gamma)$  of  $\Gamma$  has two distinct segment vertices  $x$  and  $x'$  on a given segment.

Let  $\alpha = \{e_i\}_{i=1}^k$  be any path of edges from  $x$  to  $x'$  in  $c(\Gamma)$ . Let  $1 \leq j \leq k$  be such that  $b(e_j)$  contains a segment vertex  $y$  on a segment different than the segment on which  $x, x'$  lie. Such a  $j$  must exist by definition of  $\mathcal{HD}$ . If  $b(e_j) = \{y, v\}$  and  $b(e_{j+1}) = \{y, v'\}$ , then  $v = v'$  implies  $y$  could be avoided by removing  $e_j, e_{j+1}$  from our path. Hence  $v \neq v'$ , and both are free vertices by Proposition 4.10. But our observation at the beginning of the proof shows there must exist a path between  $v$  and  $v'$  which avoids  $y$ . We can similarly eliminate any other segment vertex encountered along the way, producing a path between  $x$  and  $x'$  which does not pass through any other segment vertices.  $\square$

**4.2.4. Bundles of compactified configuration spaces from vertices and edges of a diagram.** We now describe the construction of bundles over  $\mathcal{L}_m^n$  and  $\mathcal{H}_m^n$  using the grafts of a diagram.

**Proposition 4.13.** *Let  $\Gamma \in \mathcal{LD}$  be a diagram with  $i_j$  segment vertices on the  $j$ th segment, and let  $c(\Gamma)$  be a graft of  $\Gamma$  with  $d_j$  segment vertices on the  $j$ th segment for all  $j = 1$  to  $m$ . Then  $c(\Gamma)$  gives rise to an evaluation map*

$$ev_{c(\Gamma)} : \mathcal{L}_m^n \times \prod_{j=1}^m C[i_j, \mathbb{R}] \longrightarrow C \left[ \sum_j d_j, \mathbb{R}^n \right]$$

*If  $c_1(\Gamma), \dots, c_k(\Gamma)$  are the grafts of  $\Gamma$  ordered as described above, and  $c_l(\Gamma)$  has  $d_{l,j}$  segment vertices on the  $j$ th segment for  $l = 1$  to  $k$ , then we have an evaluation map*

$$(20) \quad ev_{gr(\Gamma)} : \mathcal{L}_m^n \times \prod_{j=1}^m C[i_j, \mathbb{R}] \longrightarrow \prod_{l=1}^k C \left[ \sum_j d_{l,j}, \mathbb{R}^n \right],$$

*where  $ev_{gr(\Gamma)} = (ev_{c_1(\Gamma)}, \dots, ev_{c_k(\Gamma)})$ . Moreover, if  $\Gamma \in \mathcal{HD}$ , then we have an evaluation map*

$$ev_{gr(\Gamma)} : \mathcal{H}_m^n \times \prod_{j=1}^m C[i_j, \mathbb{R}] \longrightarrow \prod_{l=1}^k C \left[ \sum_j d_{l,j}, \mathbb{R}^n \right]$$

*whose restriction to  $\mathcal{L}_m^n \times \prod_{j=1}^m C[i_j, \mathbb{R}]$  is equal to the map in equation (20), and whose image in each factor lies in the open configuration space  $C \left( \sum_j d_{l,j}, \mathbb{R}^n \right)$ .*

*Proof.* This follows immediately from Proposition 4.12, since there is at most one segment vertex on each segment of a graft  $c(\Gamma)$ , and since homotopy links send points in distinct segments to distinct points, so that the codomain of the evaluation map is correctly identified.  $\square$

If  $\Gamma \in \mathcal{LD}$ , we now have a different evaluation maps associated with a link diagram, and this gives rise to a new way to build a bundle associated with a diagram.

**Definition 4.14.** Let  $\Gamma \in \mathcal{LD}$  be a link diagram with grafts  $c_1(\Gamma), \dots, c_k(\Gamma)$  such that  $c_l(\Gamma)$  has  $d_{l,j}$  segment vertices on the  $j$ th segment and  $s_l$  free vertices for  $l = 1$  to  $k$ . Let  $\vec{d}_l = (d_{l,1}, \dots, d_{l,m})$ . Define

$$\oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)]$$

as the pullback

$$(21) \quad \begin{array}{ccc} \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] & \longrightarrow & \prod_{l=1}^k C \left[ \sum_j d_{l,j} + s_l, \mathbb{R}^n \right] \\ \downarrow & & \downarrow pr \\ \mathcal{L}_m^n \times \prod_{j=1}^m C[i_j, \mathbb{R}] & \xrightarrow{ev_{gr(\Gamma)}} & \prod_{l=1}^k C \left[ \sum_j d_{l,j}, \mathbb{R}^n \right]. \end{array}$$

Similarly we define  $\oplus_l C[\vec{d}_l + s_l; \mathcal{H}_m^n, c_l(\Gamma)]$  when  $\Gamma \in \mathcal{HD}$  and  $\mathcal{H}_m^n$  replaces  $\mathcal{L}_m^n$ .

The notation is meant to observe that given a collection of spaces and maps  $X \rightarrow Y_i \leftarrow Z_i$  such that  $P_i$  is the pullback of this diagram for each index  $i$ , then the pullback of  $X \rightarrow \prod_i Y_i \leftarrow \prod_i Z_i$  is the product of the  $P_i$  pulled back via the diagonal map  $\Delta: X \rightarrow \prod_i X$ .

**Proposition 4.15.** *Let  $\Gamma \in \mathcal{LD}$  be a link diagram with grafts  $c_1(\Gamma), \dots, c_k(\Gamma)$  such that  $c_l(\Gamma)$  has  $d_{l,j}$  segment vertices on the  $j$ th segment for  $l = 1$  to  $k$ ,  $j = 1$  to  $m$ . Then the projection*

$$\pi_{\mathcal{L},\Gamma}: \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] \longrightarrow \mathcal{L}_m^n$$

*is a fiber bundle whose fibers are smooth finite-dimensional manifolds with corners. Moreover, if  $\Gamma \in \mathcal{HD}$ , then the projection*

$$\pi_{\mathcal{H},\Gamma}: \oplus_l C[\vec{d}_l + s_l; \mathcal{H}_m^n, c_l(\Gamma)] \longrightarrow \mathcal{H}_m^n$$

*is also a fiber bundle whose fibers are smooth finite-dimensional manifolds with corners, and*

$$(22) \quad \begin{array}{ccc} \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] & \longrightarrow & \oplus_l C[\vec{d}_l + s_l; \mathcal{H}_m^n, c_l(\Gamma)] \\ \pi_{\mathcal{L},\Gamma} \downarrow & & \downarrow \pi_{\mathcal{H},\Gamma} \\ \mathcal{L}_m^n & \longrightarrow & \mathcal{H}_m^n \end{array}$$

*is a pullback square.*

*Proof.* The projection  $\pi_{\mathcal{L},\Gamma}$  is a bundle for the same reason that Proposition 4.4 is. For  $\pi_{\mathcal{L},\Gamma}$ , this is just an extension of the observation made in (19). Lastly, the fact that the square (22) is a pullback follows directly from the definitions.  $\square$

We will denote the fibers of  $\pi_{\mathcal{L},\Gamma}$  and  $\pi_{\mathcal{H},\Gamma}$  over a link  $L \in \mathcal{L}_m^n$  or a homotopy link  $H \in \mathcal{H}_m^n$ , respectively, by

$$\pi_{\mathcal{L},\Gamma}^{-1}(L) = \oplus_l C[\vec{d}_l + s_l; L, c_l(\Gamma)]$$

and

$$\pi_{\mathcal{H},\Gamma}^{-1}(H) = \oplus_l C[\vec{d}_l + s_l; H, c_l(\Gamma)].$$

**Example 4.16.** Consider the two different evaluation maps, one from equation (15) and the other from equation (20), for the diagram  $\Gamma$  from Figure 15. For conciseness, we have omitted the compactification coordinates.

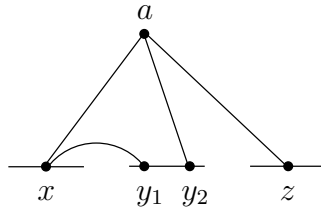


FIGURE 15.

On the one hand, using equation (15), we have

$$ev_\Gamma: \mathcal{L}_3^n \times C[1, \mathbb{R}] \times C[2, \mathbb{R}] \times C[1, \mathbb{R}] \longrightarrow C[4, \mathbb{R}^n]$$

given by

$$(L, x, y_1, y_2, z) \mapsto (L(x), L(y_1), L(y_2), L(z))$$

whose image lies in a compact subset of the subspace of all  $(w_1, w_2, w_3, w_4)$  where  $w_1 \neq w_2, w_3, w_4$ , and  $w_2, w_3 \neq w_4$  of  $(\mathbb{R}^n)^4$ . We also have the projection map

$$pr : C[5, \mathbb{R}^n] \rightarrow C[4, \mathbb{R}^n]$$

which sends  $(w_1, w_2, w_3, w_4, w_5)$  to  $(w_1, w_2, w_3, w_4)$ , so that the fibers of the bundle  $\pi_{\mathcal{L}, \Gamma} : C[(1, 2, 1) + 1; \mathcal{L}_3^n] \rightarrow \mathcal{L}_3^n$  are a subspace of  $C[5, \mathbb{R}^n]$ . The five configuration points correspond with the vertices of  $\Gamma$ , and we compactify along all diagonals of  $(\mathbb{R}^n)^5$ . Note that the bundle obtained is exactly the same for any diagram with the same vertices as  $\Gamma$ .

On the other hand,  $\Gamma$  has two graft components, one of which is the diagram with a single chord from  $x$  to  $y_1$ , and the other of which is the “tripod” with free vertex  $a$  and edges between it and  $x, y_2$ , and  $z$ . Then equation (20) gives another evaluation map

$$ev_{gr(\Gamma)} : \mathcal{L}_3^n \times C[1, \mathbb{R}] \times C[2, \mathbb{R}] \times C[1, \mathbb{R}] \longrightarrow C[2, \mathbb{R}^n] \times C[3, \mathbb{R}^n]$$

given by

$$(L, x, y_1, y_2, z) \mapsto (L(x), L(y_1), L(x), L(y_2), L(z))$$

and the image in each factor lies in the open configuration space. To build the bundle, we use the product of two projection maps

$$C[2, \mathbb{R}^n] \times C[4, \mathbb{R}^n] \longrightarrow C[2, \mathbb{R}^n] \times C[3, \mathbb{R}^n]$$

given by

$$(u_1, u_2, w_1, w_2, w_3, w_4) \mapsto (u_1, u_2, w_1, w_2, w_3)$$

to form a bundle  $\pi_{\mathcal{L}, \Gamma} : C[(1, 1, 0); \mathcal{L}_3^n] \oplus C[(1, 1, 1) + 1; \mathcal{L}_3^n] \rightarrow \mathcal{L}_3^n$ . The fibers of this bundle are isomorphic to a subspace of  $(\mathbb{R}^n)^5$ , namely the subspace of all tuples  $(w_1, w_2, w_3, w_4, w_5) = (L(x), L(y_1), L(y_2), L(z), a)$ , but  $w_3 = w_5$  is now allowed and we do not compactify along this diagonal. This is because there is no mixed edge between the free vertex  $a$  and the segment vertex  $y_1$ . We also do not compactify along  $w_2 = w_3$ . Thus the fibers are a subspace of a (compactified) partial configuration space, because not all diagonals have been removed from  $(\mathbb{R}^n)^5$ .

In general, the difference between the pullback bundle based on vertices only and the one based on vertices and edges is precisely what we saw in the last example. In the latter, the configuration space is not compactified along all the diagonals but only along those for which there is an edge. Thus if there is no edge between two vertices, the corresponding configuration points can pass through each other without the direction of collision being recorded.

**4.3. Pullback of differential forms to new bundles of configuration spaces.** For the sake of concreteness, it is necessary to choose coordinates on our configuration spaces so that we may explicitly define the pullback of forms. As the configuration space is a subspace of a product of Euclidean spaces, it will suffice instead to consider coordinate systems on such spaces.

Given a finite ordered set  $S$ , we have a unique order-preserving isomorphism  $\text{pos} : S \rightarrow \{1, \dots, |S|\}$ . For a coordinate system  $(x_1, \dots, x_{|S|})$  on  $(\mathbb{R}^n)^{|S|}$ , this gives a natural way to associate  $s \in S$  with the coordinate  $x_{\text{pos}(s)}$ .

Now suppose we have a family of subsets  $T_1, \dots, T_k$  of  $S$  whose union is equal to  $S$ . Note that each inclusion  $T_i \rightarrow S$  gives rise to a projection  $p_i : (\mathbb{R}^n)^{|S|} \rightarrow (\mathbb{R}^n)^{|T_i|}$  which projects off the coordinates associated with  $S - T_i$ . For each subset  $R$  of  $\{1, \dots, k\}$  we have the



set  $T_R = \cap_{i \in R} T_i$ , and for each inclusion  $R \rightarrow R'$  an inclusion map  $T_{R'} \rightarrow T_R$  (note the contravariance), and hence a  $k$ -dimensional cube  $R \mapsto T_R$ . The association  $R \mapsto (\mathbb{R}^n)^{|T_R|}$  is also functorial, as an inclusion  $R' \rightarrow R$  gives rise to a projection map  $(\mathbb{R}^n)^{|T_{R'}|} \rightarrow (\mathbb{R}^n)^{|T_R|}$ . Since  $S$  is the union of all the  $T_i$ , we have that  $\lim_{R \neq \emptyset} (\mathbb{R}^n)^{|T_R|} \cong (\mathbb{R}^n)^{|S|}$ . The particular isomorphism we have in mind is the one which makes the following diagram commute

$$\begin{array}{ccc} \lim_{R \neq \emptyset} (\mathbb{R}^n)^{|T_R|} & \xrightarrow{\quad} & (\mathbb{R}^n)^{|S|} \\ & \searrow & \downarrow (p_1, \dots, p_k) \\ & & \prod_{i=1}^k (\mathbb{R}^n)^{|T_i|} \end{array}$$

The diagonal arrow is the natural inclusion of the limit into the product, and the top arrow is the isomorphism we spoke of above, and we use it to give coordinates on the limit. Given a diagram  $\Gamma \in \mathcal{LD}$ , the situation described above arises with  $S = V(\Gamma)$  and the  $T_i$  as the vertices of the grafts (recall that the set of grafts is naturally ordered).

**Definition 4.17.** Let  $\Gamma \in \mathcal{LD}$  be a diagram with  $i_j$  segment vertices on the  $j$ th segment and  $s$  free vertices. Let  $e \in E(\Gamma)$ , and suppose  $b(e) = \{v, w\}$ .

- If  $v \neq w$ , then if  $e$  is oriented from  $v$  to  $w$  (or if it is not oriented, then if  $v < w$  in the ordering of the vertex set), define

$$\begin{aligned} \phi'_e: \prod_{l=1}^k C \left[ \sum_j d_{l,j} + s_l, \mathbb{R}^n \right] &\longrightarrow S^{n-1} \\ \vec{x} &\longmapsto \frac{x_{\text{pos}(w)} - x_{\text{pos}(v)}}{|x_{\text{pos}(w)} - x_{\text{pos}(v)}|}, \end{aligned}$$

and define

$$\phi_e: \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] \longrightarrow S^{n-1}$$

to be the pullback of  $\phi'_e$  under the map  $\oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] \rightarrow \prod_{l=1}^k C[\sum_j d_{l,j} + s_l, \mathbb{R}^n]$ .

- If  $v = w$ , then necessarily  $e$  joins a segment vertex with itself, and if it is oriented by the injection which sends  $v$  to 1 (or is not oriented at all),

$$\phi_e(\vec{x}, L) = D_z L(u) / |D_z L(u)|$$

where  $z$  is the point in one of the strands such that  $L(z) = x_{\text{pos}(v)}$  and  $u$  is the positive unit tangent vector to the strand at  $z$ . If  $e$  is oriented by the injection  $b(e) \rightarrow \{\pm 1\}$  which sends  $v$  to  $-1$ , then

$$\phi_e(\vec{x}, L) = -D_z L(u) / |D_z L(u)|.$$

with  $z, u$  as above.

Note that  $D_z L(u) \neq 0$  since  $L$  is an embedding; in the case of homotopy string links, which may not be embeddings, we don't have to worry about this because loops cannot be present in diagrams in  $\mathcal{HD}$ .

**Definition 4.18.** Given  $\Gamma \in \mathcal{LD}$  as above, define

$$\phi_\Gamma: \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] \longrightarrow S^{(n-1)|E(\Gamma)|}$$

by

$$\phi_\Gamma = \left( \phi_{e_1}, \dots, \phi_{e_{|E(\Gamma)|}} \right),$$

where  $\text{pos}(e_i) = i$  if the edge set is ordered, and otherwise order them according to the dictionary ordering on  $\{b(e_i)\}$  (which can be imposed since diagrams with more than one edge joining a pair of vertices are set to zero).

Let  $\text{sym}_{S^{n-1}}$  be a symmetric (its values on antipodal points are equal), smooth, unit volume top form on  $S^{n-1}$  (in Section 5.1, when we discuss the case of links in dimension 3, we will also require this form to be the unique rotation-invariant unit volume form) and let

$$\omega = \bigwedge_{|E(\Gamma)|} \text{sym}_{S^{n-1}}$$

Finally consider the pullback form

$$\alpha_\Gamma = (\phi_\Gamma)^* \omega \in \Omega^{(n-1)(|E(\Gamma)|)} \left( \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] \right).$$

Notice that nothing changes in the case of homotopy links. For a diagram  $\Gamma \in \mathcal{HD}$ , we again use edges (but there are no longer any loops) to pull back a product of forms  $\omega$  from  $S^{(n-1)|E(\Gamma)|}$  to the space  $\oplus_l C[\vec{d}_l + s_l; \mathcal{H}_m^n, c_l(\Gamma)]$ , although we will write  $\alpha_\Gamma^{\mathcal{H}}$  for the pullback form when  $\Gamma \in \mathcal{HD}$ .

Observe also that the same definitions are valid for the bundle  $C[\vec{i} + s; \mathcal{L}_m^n, \Gamma]$ . Namely, we have a map

$$\bar{\phi}_\Gamma: C[\vec{i} + s; \mathcal{L}_m^n, \Gamma] \longrightarrow S^{(n-1)|E(\Gamma)|}$$

dictated by the edges of  $\Gamma$ , and this can be used for pulling back a product of volume forms to give a form  $\bar{\alpha}_\Gamma = (\bar{\phi}_\Gamma)^* \omega$ . This case was considered in [27, Section 3.2].

**4.4. Configuration space integrals of string links and homotopy string links.** We are finally ready to produce forms on spaces of links and homotopy links. Namely, the form  $\alpha_\Gamma$  can be pushed forward, or integrated along the fiber of the bundle

$$\pi_{\mathcal{L}, \Gamma}: \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] \longrightarrow \mathcal{L}_m^n$$

to produce a form  $(\pi_{\mathcal{L}, \Gamma})_* \alpha_\Gamma$ , or, as we will usually denote it, a form

$$(I_{\mathcal{L}})_\Gamma \in \Omega^{(n-1)|E(\Gamma)| - n|V(\Gamma)_{free}| - |V(\Gamma)_{seg}|}(\mathcal{L}_m^n).$$

The value of this form on a link  $L \in \mathcal{L}_m^n$  is thus

$$(I_{\mathcal{L}})_\Gamma(L) = \int_{\pi_{\mathcal{L}, \Gamma}^{-1}(L) = \oplus_l C[\vec{d}_l + s_l; L, c_l(\Gamma)]} \alpha_\Gamma.$$

The degree  $(n-1)|E(\Gamma)| - n|V(\Gamma)_{free}| - |V(\Gamma)_{seg}|$  of  $(I_{\mathcal{L}})_{\Gamma}$  is the difference of the degree of  $\alpha_{\Gamma}$  and the dimension of the fiber  $\pi_{\mathcal{L},\Gamma}^{-1}(L)$ . The difference is equal to  $(n-3)(|E(\Gamma)| - |V(\Gamma)_{free}|) + d$ , where  $d = \deg(\Gamma)$ , so that we have constructed a map

$$(23) \quad I_{\mathcal{L}}: \mathcal{LD}^d \longrightarrow \Omega^{(n-3)(|E(\Gamma)| - |V(\Gamma)_{free}|) + d}(\mathcal{L}_m^n).$$

For a diagram  $\Gamma \in \mathcal{HD}$ , we integrate the associated form  $\alpha_{\Gamma}^{\mathcal{H}}$  along the bundle

$$\pi_{\mathcal{H},\Gamma}: \oplus_l C[\vec{d}_l + s_l; \mathcal{H}_m^n, c_l(\Gamma)] \longrightarrow \mathcal{H}_m^n.$$

This gives a form

$$(I_{\mathcal{H}})_{\Gamma} \in \Omega^{(n-1)|E(\Gamma)| - n|V(\Gamma)_{free}| - |V(\Gamma)_{seg}|}(\mathcal{H}_m^n)$$

whose value on a homotopy link  $H \in \mathcal{H}_m^n$  is

$$(I_{\mathcal{H}})_{\Gamma}(H) = \int_{\pi_{\mathcal{H},\Gamma}^{-1}(H) = \oplus_l C[\vec{d}_l + s_l; H, c_l(\Gamma)]} \alpha_{\Gamma}^{\mathcal{H}}.$$

Again rewriting the degree of the form, we thus have a map

$$(24) \quad I_{\mathcal{H}}: \mathcal{HD}^d \longrightarrow \Omega^{(n-3)(|E(\Gamma)| - |V(\Gamma)_{free}|) + d}(\mathcal{H}_m^n).$$

*Remark 4.19.* It is immediate from the definition that maps  $I_{\mathcal{L}}$  and  $I_{\mathcal{H}}$  are also compatible with the inclusion

$$\mathcal{L}_m^n \hookrightarrow \mathcal{H}_m^n.$$

That is, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{HD} & \xrightarrow{\quad} & \mathcal{LD} \\ I_{\mathcal{H}} \downarrow & & \downarrow I_{\mathcal{L}} \\ \Omega^*(\mathcal{H}_m^n) & \longrightarrow & \Omega^*(\mathcal{L}_m^n) \end{array}$$

This is precisely what we were after when we refined the definition of the bundles we integrate over.

Now note that again nothing changes for the case of the pullback bundle defined without consideration of the grafts. Namely, the construction of  $(\pi_{\mathcal{L},\Gamma})_* \alpha_{\Gamma}$  goes through exactly the same way to give a form  $(\overline{\pi}_{\mathcal{L},\Gamma})_* \overline{\alpha}_{\Gamma}$  by pushing forward the form  $\overline{\alpha}_{\Gamma}$  along the map

$$\overline{\pi}_{\mathcal{L},\Gamma}: C[\vec{i} + s; \mathcal{L}_m^n, \Gamma] \longrightarrow \mathcal{L}_m^n$$

from Proposition 4.4. We now want to show that the forms we obtain by integrating along this bundle are the same as the forms we obtain by integrating along

$$\pi_{\mathcal{L},\Gamma}: \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] \longrightarrow \mathcal{L}_m^n$$

are the same as in the case of integration along the bundle

$$\overline{\pi}_{\mathcal{L},\Gamma}: C[\vec{i} + s; \mathcal{L}_m^n, \Gamma] \longrightarrow \mathcal{L}_m^n.$$

This will finally show that our way of setting up configuration space integrals for links is indeed a refinement of the way that has been considered in literature thus far.

**Proposition 4.20.** *For any  $\Gamma \in \mathcal{LD}$ ,  $(\pi_{\mathcal{L},\Gamma})_* \alpha_{\Gamma} = (\overline{\pi}_{\mathcal{L},\Gamma})_* \overline{\alpha}_{\Gamma}$ .*

*Proof.* The map between fibers is the inclusion of an open dense set. Namely, the two fibers are the same on the biggest stratum, namely the open configuration space. They differ in that  $\pi_{\mathcal{L},\Gamma}^{-1}(L)$  has more diagonals of  $\mathbb{R}^{n|V(\Gamma)|}$  removed and compactified. However, the difference between the two is at least of codimension 1 and so the integrals are the same.  $\square$

We next give a few examples of these configuration space integrals.

**Example 4.21** (Diagrams with no free vertices). One special case is that of diagrams with no free vertices, i.e. those that only contain chords and loops. In that case, the construction simplifies since there are no pullback constructions as in Definition 4.14, and the bundles constructed are trivial. For example, if  $\Gamma \in \mathcal{LD}$  is the diagram from Figure 16 (where we have omitted the edge orientations and labels for simplicity), then the map  $\phi_\Gamma$  is a composition

$$\phi_\Gamma: \mathcal{L}_3^n \times C[3, \mathbb{R}] \times C[1, \mathbb{R}] \times C[2, \mathbb{R}] \xrightarrow{ev_\Gamma} C[2, \mathbb{R}^n]^4 \times C[1, \mathbb{R}^n] \longrightarrow (S^{(n-1)})^5$$

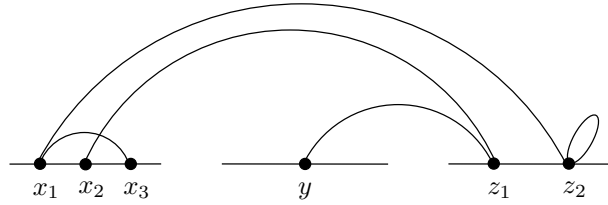


FIGURE 16.

After pulling back the product of five symmetric top forms from  $(S^{(n-1)})^5$ , the integration takes place along the trivial bundle

$$\pi_{\mathcal{L},\Gamma}: \mathcal{L}_3^n \times C[3, \mathbb{R}] \times C[1, \mathbb{R}] \times C[2, \mathbb{R}] \longrightarrow \mathcal{L}_3^n.$$

**Example 4.22** (Linking number). Another special case, and in fact the case that motivated Bott and Taubes to define configuration space integrals for knots in [4], is that of the linking number of a two-component link in  $\mathbb{R}^3$ . Namely, suppose  $\Gamma$  is the diagram with a single chord between segments  $i$  and  $j$  and no free vertices or segment vertices on other segments, as in Figure 17.

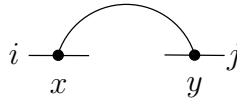


FIGURE 17.

Then the integration described above recovers the classical Gauss integral computing the linking number of strands  $i$  and  $j$  of a link or a homotopy link  $L$ , which we will denote by  $\text{lk}(L_i, L_j)$ . In short,

$$\text{lk}(L_i, L_j) = (I_{\mathcal{H}})_\Gamma(L) = (I_{\mathcal{L}})_\Gamma(L) = \int_{C[1, \mathbb{R}^3] \times C[1, \mathbb{R}^3] \cong \mathbb{R}^3 \times \mathbb{R}^3} \left( \frac{L(x) - L(y)}{|L(x) - L(y)|} \right)^* \text{sym}_{S^2}.$$

To see how shuffle products of integrals give products of linking numbers, see Example 4.26.

**Example 4.23** (Homotopy links with one strand). Consider the case of  $\mathcal{H}_1^n$ ,  $n \geq 3$ . Now the only diagram in  $\mathcal{HD}$  is the empty diagram, and so the integration does not produce any forms in this case. This is of course consistent with the fact that  $\mathcal{H}_1^n$ ,  $n \geq 3$ , is a contractible space (Corollary 2.4).

**4.5. Integration is a map of differential algebras.** The goal of this section is to prove Theorem 4.28, which says that integration along the fiber is a map of differential algebras. This theorem will follow from Propositions 4.24–4.27. Most of the statements follow easily from the case of knots considered in [5, 6], but for completeness and convenience of the reader, we give fairly complete outlines of their proofs. We elaborate on the fact that  $I_{\mathcal{L}}$  is a map of algebras; this result is stated in [6] but without justification. In addition, we also observe that the same proofs apply for the case of the map  $I_{\mathcal{H}}$ , and that in fact some of the results now even work for  $n = 3$ .

We begin with

**Proposition 4.24.** *For  $n \geq 3$  and  $m \geq 1$ ,  $I_{\mathcal{L}}$  and  $I_{\mathcal{H}}$  are well-defined homomorphisms.*

*Proof.* We check that integration is compatible with the relations from Definition 3.10. For the first condition, if  $\Gamma$  has a double edge, then  $\phi_{\Gamma}$  factors through a product with one fewer sphere, since one direction is repeated:

$$\begin{array}{ccc} \oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)] & \xrightarrow{\phi_{\Gamma}} & S^{(n-1)|E(\Gamma)|} \\ & \searrow & \nearrow \\ & S^{(n-1)(|E(\Gamma)|-1)} & \end{array}$$

Then the pullback of  $\omega$  via  $\phi_{\Gamma}$  is the same as the pullback through the factorization. However, the dimension of  $\omega$  is greater than  $(n-1)(|E(\Gamma)|-1)$  and so the pullback is zero. The same argument holds when  $\oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)]$  is replaced by  $\oplus_l C[\vec{d}_l + s_l; \mathcal{H}_m^n, c_l(\Gamma)]$ .

The other two conditions in Definition 3.10 are in fact designed for compatibility with the integration. Namely, if  $n$  is even or odd, then switching two configuration points on the link (i.e. switching two copies of  $\mathbb{R}$ ) gives  $\oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)]$  and  $\oplus_l C[\vec{d}_l + s_l; \mathcal{H}_m^n, c_l(\Gamma)]$  different orientations and produces an integral with a different sign. A similar situation occurs if two free configuration points are switched and  $n$  is odd, and if two maps are switched in the product  $\phi_{\Gamma}$  and  $n$  is odd (this corresponds to switching the order of edges). The latter case introduces a sign because the effect is that of transposition of two even-dimensional forms. Again, a minus sign is introduced in the integral. Thus  $I_{\mathcal{L}}$  and  $I_{\mathcal{H}}$  are well-defined and they are homomorphisms since pullback of forms and integration are linear.  $\square$

**Proposition 4.25.** *For  $n \geq 3$  and  $m \geq 1$ ,  $I_{\mathcal{L}}$  and  $I_{\mathcal{H}}$  are maps of algebras.*

*Proof.* The shuffle product of diagrams from Definition 3.18 corresponds precisely to the wedge product of forms which gives the deRham complex the structure of an algebra. We have

$$(25) \quad (I_{\mathcal{L}})_{\Gamma_1 \bullet \Gamma_2} = (I_{\mathcal{L}})_{\Gamma_1} \wedge (I_{\mathcal{L}})_{\Gamma_2} \quad \text{and} \quad (I_{\mathcal{H}})_{\Gamma_1 \bullet \Gamma_2} = (I_{\mathcal{H}})_{\Gamma_1} \wedge (I_{\mathcal{H}})_{\Gamma_2}.$$

This is a direct generalization of the same statement for long knots [6, Proposition 5.3]. That result is provided without much explanation, so we elaborate on (25) a bit here.

Recall that one way to think about the wedge product is as follows:

Given a  $k$ -form  $\alpha$  and an  $l$ -form  $\beta$ , the wedge product is a multilinear  $(k+l)$ -form whose value on the variables  $x_1, \dots, x_{k+l}$  is

$$\alpha \wedge \beta(x_1, \dots, x_{k+l}) = \sum_{\sigma \in \text{Shuffle}(k,l)} \text{sign}(\sigma) \alpha(x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(k)}) \beta(x_{\sigma(k+1)} \wedge \dots \wedge x_{\sigma(k+l)}),$$

where  $\text{Shuffle}$  the subset of the permutations of  $\{1, \dots, k+l\}$  such that  $\sigma(1) < \sigma(2) < \dots < \sigma(k)$  and  $\sigma(k+1) < \sigma(k+2) < \dots < \sigma(k+l)$ .

Thus, given diagrams  $\Gamma_1$  and  $\Gamma_2$ , each shuffle  $v_{\sigma(1)}, \dots, v_{\sigma(k+l)}$  of the segment vertices on one segment corresponds to configurations on a strand of a link appearing in that order. In other words, the integration takes place over a “piece” of  $\mathbb{R}^{k+l}$  determined by  $x_{\sigma(1)} < \dots < x_{\sigma(k+l)}$  (plus as many copies of  $\mathbb{R}^n$  as there are free vertices in both diagrams, since they are free to move anywhere). Adding the integrals over all shuffles, we get  $(I_{\mathcal{L}})_{\Gamma_1 \bullet \Gamma_2}$ , and in this sum, integration thus takes places over all pieces of  $\mathbb{R}^{k+l}$ .<sup>1</sup> The integrals agree on the boundary, so that this sum can be represented by a single integral, taken over  $\mathbb{R}^{k+l}$  (again plus some copies of  $\mathbb{R}^n$ ). But this integral is a product of integrals by Fubini’s Theorem, one taken over  $\mathbb{R}^k$  and one over  $\mathbb{R}^l$  (plus as many copies of  $\mathbb{R}^n$  in each as there are free vertices in the two diagrams whose shuffle product was taken). This product of integrals is precisely  $(I_{\mathcal{L}})_{\Gamma_1} \wedge (I_{\mathcal{L}})_{\Gamma_2}$ . The same is true when  $I_{\mathcal{L}}$  is replaced by  $I_{\mathcal{H}}$ .  $\square$

Below is an example of the argument given in the previous proposition.

**Example 4.26.** Recalling Example 4.22, we now also see from Proposition 4.25 how shuffle products of diagrams, each with one chord between different strands, corresponds to the powers and products of linking numbers. For example, if  $\Gamma_1$  and  $\Gamma_2$  are as in Figure 18, then their shuffle product is given in Figure 19.

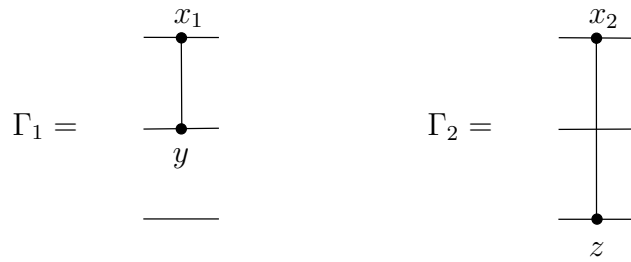


FIGURE 18.

<sup>1</sup>This is much like what happens in the Eilenberg-Zilber map.

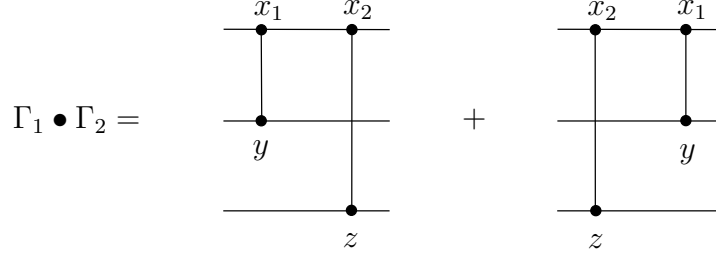


FIGURE 19.

This sum is given by

$$\begin{aligned}
(I_{\mathcal{L}})_{\Gamma_1 \bullet \Gamma_2}(L) &= \\
(I_{\mathcal{H}})_{\Gamma_1 \bullet \Gamma_2}(L) &= \int_{\substack{-\infty \leq x_1 \leq x_2 \leq \infty \\ y \in \mathbb{R} \\ z \in \mathbb{R}}} \left( \frac{L(x_1) - L(y)}{|L(x_1) - L(y)|} \right)^* \text{sym}_{S^2} \wedge \left( \frac{L(x_2) - L(z)}{|L(x_2) - L(z)|} \right)^* \text{sym}_{S^2} \\
&+ \int_{\substack{-\infty \leq x_2 \leq x_1 \leq \infty \\ y \in \mathbb{R} \\ z \in \mathbb{R}}} \left( \frac{L(x_2) - L(z)}{|L(x_2) - L(z)|} \right)^* \text{sym}_{S^2} \wedge \left( \frac{L(x_1) - L(y)}{|L(x_1) - L(y)|} \right)^* \text{sym}_{S^2} \\
&\stackrel{(i)}{=} \int_{\substack{-\infty \leq x_1 \leq x_2 \leq \infty \\ y \in \mathbb{R} \\ z \in \mathbb{R}}} \left( \frac{L(x_1) - L(y)}{|L(x_1) - L(y)|} \right)^* \text{sym}_{S^2} \wedge \left( \frac{L(x_2) - L(z)}{|L(x_2) - L(z)|} \right)^* \text{sym}_{S^2} \\
&+ \int_{\substack{-\infty \leq x_2 \leq x_1 \leq \infty \\ y \in \mathbb{R} \\ z \in \mathbb{R}}} \left( \frac{L(x_1) - L(y)}{|L(x_1) - L(y)|} \right)^* \text{sym}_{S^2} \wedge \left( \frac{L(x_2) - L(z)}{|L(x_2) - L(z)|} \right)^* \text{sym}_{S^2} \\
&\stackrel{(ii)}{=} \int_{\substack{(x_1, x_2) \in \mathbb{R}^2 \\ y \in \mathbb{R} \\ z \in \mathbb{R}}} \left( \frac{L(x_1) - L(y)}{|L(x_1) - L(y)|} \right)^* \text{sym}_{S^2} \wedge \left( \frac{L(x_2) - L(z)}{|L(x_2) - L(z)|} \right)^* \text{sym}_{S^2} \\
&\stackrel{(iii)}{=} \int_{\substack{x_1 \in \mathbb{R} \\ y \in \mathbb{R}}} \left( \frac{L(x_1) - L(y)}{|L(x_1) - L(y)|} \right)^* \text{sym}_{S^2} \cdot \int_{\substack{x_2 \in \mathbb{R} \\ z \in \mathbb{R}}} \left( \frac{L(x_2) - L(z)}{|L(x_2) - L(z)|} \right)^* \text{sym}_{S^2} \\
&= \text{lk}(L_1, L_2) \cdot \text{lk}(L_1, L_3).
\end{aligned}$$

The equality (i) is true because switching the order of the maps, and hence pullbacks, does not matter ( $n = 3$  here, so it is odd). Equation (ii) is true because the two integrals agree on the boundary  $x_1 = x_2$ . The diagram representing this boundary in both cases is the one in Figure 20.



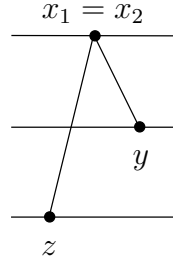


FIGURE 20.

The integrals, however, come with opposite signs on the boundary because of the orientation in the Fulton-MacPherson compactification (the compactification is the reason we write “ $\leq$ ” everywhere). One should also pay attention to the boundary contributions along faces at infinity, but the integrals along all those vanish (for example, if  $x_1$  goes to  $-\infty$ , then the map giving the direction between  $x_1$  and  $y$  is constant and the pullback of the volume form then must be zero; arguments are similar for other faces at infinity).

Lastly, note that in the expression following equality (iii), we get the ordinary product of integrals, rather than a wedge product, since the forms we obtain are 0-forms, i.e. functions on  $\mathcal{L}_3^3$  (or  $\mathcal{H}_3^3$ ), and the wedge product in that case is the usual product.

**Proposition 4.27.** *For  $n \geq 4$  and  $m \geq 1$ ,  $I_{\mathcal{L}}$  is a map of differential complexes. For  $n \geq 3$  and  $m \geq 1$ , the same is true  $I_{\mathcal{H}}$ .*

*Proof.* For the case of links, this is Theorem 3.6 in [27]. That statement, in turn, follows directly from the same result for knots established in [5]. In short, Stokes’ Theorem for manifolds with corners states that

$$(26) \quad d((\pi_{\mathcal{L},\Gamma})_*\alpha_{\Gamma}) = (\pi_{\mathcal{L},\Gamma})_*d\alpha_{\Gamma} + (\partial\pi_{\mathcal{L},\Gamma})_*\alpha_{\Gamma} = (\partial\pi_{\mathcal{L},\Gamma})_*\alpha_{\Gamma}.$$

The last equality is true because  $\alpha_{\Gamma}$  is a closed form (since it is a pullback of a closed form, namely the product of volume forms on the sphere) so  $d\alpha_{\Gamma} = 0$ . The term  $(\partial\pi_{\mathcal{L},\Gamma})_*\alpha_{\Gamma}$  denotes the sum of integrals along all codimension one faces of  $\oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)]$ . The faces given by two points colliding, called *principal*, correspond to contractions of edges in  $\mathcal{LD}$ . To get a map of complexes, therefore, it remains to show the vanishing of the restriction of the integral to all other faces. Recalling the discussion following Definition 4.2, such faces are characterized by more than two points coming together at the same time or one or more points escaping to infinity. The former are called *hidden faces*, and the latter are called *faces at infinity*.

The vanishing arguments depend on the various cases. Some are combinatorial, but most depend on dimension-counting. A representative argument is given in the proof of Proposition 4.24. For the case of knots, the arguments can be found in [5, 26], but, as noted in [27, Theorem 3.6], the generalization to links is immediate. Importantly, since  $\mathcal{HD}$  is a subcomplex of  $\mathcal{LD}$ , it is also immediate that all the vanishing arguments go through exactly the same way for the case of homotopy links and the map  $(\pi_{\mathcal{H},\Gamma})_*\alpha_{\Gamma}^{\mathcal{H}}$ .

As will be discussed in Section 5.1, the argument that shows the vanishing along the hidden face where all configuration points come together (so-called *anomalous face*) does not work for

$n = 3$  so that one does not get a chain map in the case of  $\mathcal{L}_m^3$ . Note that the collision of all configuration points can only happen if all the segment vertices in a diagram  $\Gamma \in \mathcal{LD}$  are concentrated on one segment (see Remark 5.1 for more on this). However, it is immediate from the definition of  $\mathcal{HD}$  that no  $\Gamma \in \mathcal{HD}$  can have all its segment vertices on one segment (unless  $\Gamma$  is the empty diagram). Therefore one never encounters an anomalous face in the case of  $I_{\mathcal{H}}$  and we thus also get a chain map in the case of homotopy links even for  $n = 3$ .  $\square$

Putting together the previous three Propositions, we obtain

**Theorem 4.28.** *For  $n \geq 4$  and  $m \geq 1$ , the integration map*

$$(27) \quad I_{\mathcal{L}}: \mathcal{LD}^d \longrightarrow \Omega^{(n-3)(|E(\Gamma)| - |V(\Gamma)_{free}|) + d}(\mathcal{L}_m^n)$$

$$\Gamma \longmapsto \left( L \longmapsto (I_{\mathcal{L}})_{\Gamma}(L) = \int_{\pi_{\mathcal{L},\Gamma}^{-1}(L) = \oplus_l C[\vec{d}_l + s_l; L, c_l(\Gamma)]} \alpha_{\Gamma} \right)$$

*is a morphism of differential algebras. For  $n \geq 3$  and  $m \geq 1$ , the same is true for the map*

$$(28) \quad I_{\mathcal{H}}: \mathcal{HD}^d \longrightarrow \Omega^{(n-3)(|E(\Gamma)| - |V(\Gamma)_{free}|) + d}(\mathcal{H}_m^n)$$

$$\Gamma \longmapsto \left( H \longmapsto (I_{\mathcal{H}})_{\Gamma}(H) = \int_{\pi_{\mathcal{H},\Gamma}^{-1}(H) = \oplus_l C[\vec{d}_l + s_l; H, c_l(\Gamma)]} \alpha_{\Gamma}^{\mathcal{H}} \right).$$

*Remark 4.29.* Conjecturally, the map  $I_{\mathcal{L}}$  is a quasi-isomorphism. This is likely since it is known that  $\mathcal{LD}$  and  $\mathcal{L}_m^n$  have isomorphic cohomology. It would be interesting to examine the same question for the map  $I_{\mathcal{H}}$ .

*Remark 4.30.* Changing the form  $\text{sym}_{S^{n-1}}$  to another symmetric volume form  $\text{sym}'_{S^{n-1}}$  does not matter since the difference between the forms obtained this way is an exact form [5, Proposition 4.5].

## 5. CONFIGURATION SPACE INTEGRALS AND FINITE TYPE INVARIANTS OF HOMOTOPY STRING LINKS

In this section, we focus on classical homotopy links, so  $n = 3$ , and we want to see what invariants, i.e. forms in degree zero, one obtains through our integration. It turns out that what appears are precisely *finite type invariants* of homotopy links. The main result of this section is that  $\mathbb{R}$ -valued finite type  $k$  invariants of homotopy links correspond precisely to the vector space of weight systems  $\mathcal{HW}_k$  via configuration space integrals. That the two are isomorphic is known [3], but we exhibit this isomorphism explicitly using configuration space integrals. For links, this was done in [27, Section 4] and is for convenience restated below as Theorem 5.6. The bulk of this section is devoted to proving the same statement for homotopy links (Theorem 5.8). However, since the proofs are essentially identical for links and homotopy links, and since we supply most of the details here, this section can be thought of as also giving the proof of Theorem 5.6. Only a sketch of the proof of that theorem is given in [27]. See Remark 5.13 for more details.

One important difference between links and homotopy links in this section is that one no longer has to worry about anomalous faces in the case of homotopy links (see Remark 5.1). Looking at equation (23), we see that it is precisely diagrams in degree zero that give such degree zero forms, so this is why we considered them in Section 3.4; the reader may find it helpful to review that section before proceeding with this one.

**5.1. The anomalous correction.** As mentioned in the proof of Proposition 4.27, the map  $I_{\mathcal{L}}$  is not a chain map for  $n = 3$ . Recall that, to prove that  $I_{\mathcal{L}}$  commutes with the differential, we have to check that the restrictions of  $I_{\mathcal{L}}$  to the hidden faces or faces at infinity of  $\oplus_l C[\vec{d}_l + s_l; \mathcal{L}_m^n, c_l(\Gamma)]$  vanish (as required by Stokes' Theorem). While this indeed happens for  $n > 3$ , there is one type of a face for which this fails in the case  $n = 3$ . This is known as the *anomalous face* and is indexed by all points of a connected component of a diagram colliding at the same time. To fix this, one introduces a correction term which we give for the convenience of the reader in equation (30) below. This correction was first given by Bott and Taubes [4] in the case of knots and was generalized to links in [27, Theorem 4.5].

*Remarks 5.1.*

(1) The collision of all configuration points can only take place in the space

$$C[0, \dots, 0, k_j, 0, \dots, 0; L, \Gamma], \quad 1 \leq j \leq m,$$

because points on different strands of a link cannot come together. The diagram  $\Gamma$  which corresponds to this situation thus must have a connected component with all its segment vertices on a single segment (and does not contain chords – if it does, the integral along the anomalous face vanishes; see [27, Proposition 4.3]). Since the integral associated to such a  $\Gamma$  computes a form on the space of knots (i.e. only on the  $j$ th strand of the link), the issue with anomalous faces is thus purely a knotting phenomenon, rather than a linking one.

(2) As a consequence of the previous remark, and as was mentioned in the proof of Proposition 4.27, anomalous faces are thus not an issue for homotopy links. Because of how the complex  $\mathcal{HD}$  is defined, a homotopy link diagram concentrated on one segment must be the empty diagram. The pushforward  $\pi_{\mathcal{H}, \Gamma}$  along the anomalous face thus vanishes and this is why  $I_{\mathcal{H}}$  does not require a correction factor in Theorem 5.8 below.

To give the complete picture, we remind the reader of what the correction for the case of links is: Let  $\text{sym}_{S^2}$  now be a *rotation-invariant* smooth unit volume form on  $S^2$  and recall the definition of a connected component of a diagram (Definition 3.4). Consider the map

$$(29) \quad \tilde{I}_{\mathcal{L}}: \mathcal{LD} \longrightarrow \Omega^0(\mathcal{L}_m^3)$$

defined by:

- If  $\Gamma$ 
  - does not contain a connected component all of whose segment vertices all lie on a single segment, or;
  - does contain a connected component all of whose segment vertices are on a single segment, but this connected component contains a chord, then

$$(\tilde{I}_{\mathcal{L}})_{\Gamma}(L) = (I_{\mathcal{L}})_{\Gamma}(L);$$

- If  $\Gamma$  contains connected components  $\Gamma_1, \Gamma_2, \dots, \Gamma_l$  with no chords whose segment vertices are on segments  $s_1, s_2, \dots, s_l$ , respectively, then

$$(30) \quad (\tilde{I}_{\mathcal{L}})_{\Gamma}(L) = (I_{\mathcal{L}})_{\Gamma}(L) - \sum_{j=1}^l \mu_{\Gamma_j} \int_{C[2, L_{s_j}]} \left( \frac{x_1 - x_2}{|x_1 - x_2|} \right)^* \text{sym}_{S^2}$$

Here  $L_{s_j}$  is the  $s_j$ th strand of the link  $L$  and  $\mu_{\Gamma_j}$  are real numbers (usually difficult to determine). We then have

**Theorem 5.2.** [27, Theorem 4.5] *The restriction of  $\tilde{I}_{\mathcal{L}}$  to all hidden faces and faces at infinity is zero.*

*Remark 5.3.* We again wish to emphasize that, for this theorem to be true, it is important that we start with a rotation-invariant form  $\text{sym}_{S^2}$  on  $S^2$ . For details on why this is necessary, see [4].

**5.2. Finite type invariants and chord diagrams.** We now briefly review the theory of finite type link invariants and recall how it is connected to the combinatorics of chord diagrams. Literature on this subject is abundant, but a good start for the case of knots is [2]. For a slightly more detailed overview than we give here for the case of links, see [27, Section 4.3].

Suppose we are given a link or a homotopy link invariant  $V$ , so that  $V$  is an element of  $H^0(\mathcal{L}_m^3)$  or  $H^0(\mathcal{H}_m^3)$ . This invariant can be extended to *singular* links, by which we mean links with finitely many double-point self intersections where the two derivatives are independent. The singularities for ordinary links can come from a single strand crossing itself or two different strands intersecting. For homotopy links, we only consider those singularities arising from two different strands (if there is a singularity on a single strand, we ignore it). The extension of  $V$  is defined via the skein relation given in Figure 21. The orientation on the link, which for us is given by the natural orientation of each of the  $m$  copies of  $\mathbb{R}$ , needs to be emphasized so that the two resolutions can be distinguished from each other (otherwise the two pictures on the right side of the equation in Figure 21 can be rotated into one another).

$$V\left(\begin{array}{c} \nearrow \quad \nwarrow \\ \bullet \\ \swarrow \quad \searrow \end{array}\right) = V\left(\begin{array}{c} \nearrow \quad \nwarrow \\ \swarrow \quad \searrow \end{array}\right) - V\left(\begin{array}{c} \nearrow \quad \nwarrow \\ \nwarrow \quad \swarrow \end{array}\right)$$

FIGURE 21. Skein relation.

A  $k$ -singular link (a link with  $k$  singularities) thus produces  $2^k$  links on which  $V$  can be evaluated. We will call these the *resolutions* of a singular link. Because of the signs, the order in which singularities are resolved does not matter.

**Definition 5.4.** The invariant  $V$  is *finite type  $k$*  (or *Vassiliev of type  $k$* ) if it vanishes on links with  $k + 1$  singularities.

Let

$\mathcal{LV}_k$  = real vector space generated by finite type  $k$  link invariants;

$\mathcal{HV}_k$  = real vector space generated by finite type  $k$  homotopy link invariants.

Note that  $\mathcal{LV}_{k-1} \subset \mathcal{LV}_k$  and  $\mathcal{HV}_{k-1} \subset \mathcal{HV}_k$  so that it makes sense to form quotients  $\mathcal{LV}_k/\mathcal{LV}_{k-1}$  and  $\mathcal{HV}_k/\mathcal{HV}_{k-1}$ .

Next we want to describe a map  $f$  which to a finite type invariant associates a weight system. The construction is standard in finite type knot theory and this map is in fact the first connection between finite type invariants and the combinatorics of chord diagrams described in Section 3.4 (a detailed account of this in the case of knots is given in [2]). Here we recall and adapt it to the setting of homotopy links. The inverse of  $f$  is given precisely by configuration space integrals and this is how one obtains isomorphisms in Theorems 5.6 and 5.8 below. The former was already proven in [27] so we will only provide a proof for the latter here.

*Remark 5.5.* Another way to construct an inverse to  $f$  is the famous *Kontsevich Integral* [11]. In fact, this integral provided the first proof of the isomorphism from Theorem 5.6 in the case of knots, i.e. when  $m = 1$ . This is known as the Fundamental Theorem of Finite Type Invariants.

To define  $f$ , first recall Theorem 3.33 and the terminology introduced after its statement. Let  $\Gamma$  be a chord diagram in  $\mathcal{HC}_k^0$  and let  $H_\Gamma$  be any singular homotopy link with singularities as prescribed by  $\Gamma$ . By this we mean that  $H_\Gamma$  is any smooth map of  $m$  copies of  $\mathbb{R}$  in  $\mathbb{R}^3$  with, as usual, disjoint images and which is fixed outside a compact set, but which also has  $k$  “nice” self-intersections (locally embedded, derivatives independent at intersection point) given by  $H_\Gamma(x_i) = H_\Gamma(y_j)$ ,  $x_i, y_j \in \mathbb{R}$ , if there is a chord between vertices  $x_i$  and  $y_j$  in  $\Gamma$ . The points  $H_\Gamma(x_i)$  and  $H_\Gamma(y_j)$  are required to be on the strands corresponding to the segments that vertices  $x_i$  and  $y_j$  are on, and if  $x_i$  ( $y_j$ ) comes before some other segment vertex  $x_{i'}$  ( $y_{j'}$ ) in the ordering of the vertices of  $\Gamma$  (we picture  $x_i$  as lying to the left of  $x_{i'}$  in this case), then  $x_i < x_{i'}$  ( $y_j < y_{j'}$ ) as points in  $\mathbb{R}$  (by abuse of notation, we label the segment vertices the same way as coordinates in  $\mathbb{R}$ ). An example is given in Figure 22.

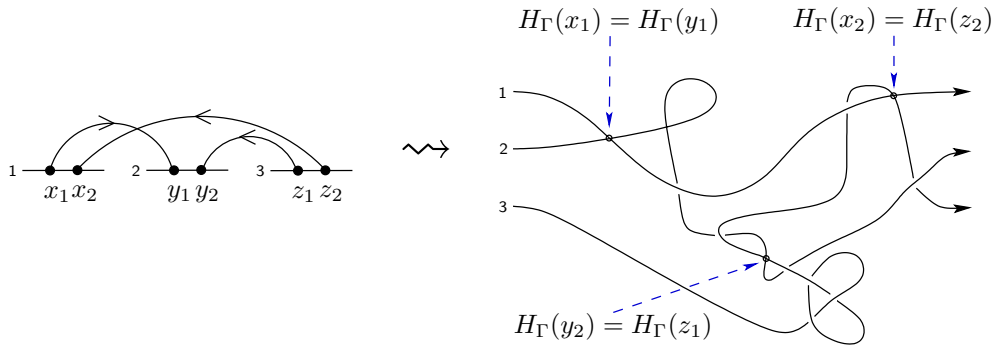


FIGURE 22. An example of a homotopy link  $H_\Gamma$  associated to a chord diagram  $\Gamma \in \mathcal{HC}_3^0$ . The only requirement is that the relative positions of the singularities respect the relative positions of the chords. Note that strand 2 intersects itself but we ignore such singularities.

Now consider the value of a type  $k$  invariant  $V \in \mathcal{HV}_k$  on (the sum of the resolutions of)  $H_\Gamma$ . This value remains unchanged if a crossing between two strands of  $H_\Gamma$  is switched because, by the skein relation,

$$V(H_\Gamma) - V(H_\Gamma \text{ with a crossing changed}) = V(\text{some } (k+1)\text{-singular link}) = 0.$$

This means that  $V$  does not depend on a particular link but only on the placement of singularities. It thus makes sense to define a map

$$f: \mathcal{HV}_k \longrightarrow \mathcal{HCW}_k$$

$$V \longmapsto \left( \begin{array}{ccc} W: \mathcal{HC}_k^0/(4T, 1T) & \longrightarrow & \mathbb{R} \\ \Gamma & \longmapsto & V(H_\Gamma) \end{array} \right)$$

It follows immediately from the definitions that the kernel of  $f$  consists precisely of type  $k-1$  invariants, so that  $f$  becomes an injection

$$(31) \quad f: \mathcal{HV}_k/\mathcal{HV}_{k-1} \hookrightarrow \mathcal{HCW}_k.$$

We can then use the isomorphism

$$\mathcal{HW}_k \cong \mathcal{HCW}_k$$

from (12) to extend  $f$  to weight systems on trivalent diagrams. Namely, recall that this isomorphism is induced by sending a chord diagram to itself and trivalent diagram to a sum of chord diagrams obtained from it by resolving all the free vertices via the STU relation. We obtain then an extension of  $f$  to an injection

$$(32) \quad f: \mathcal{HV}_k/\mathcal{HV}_{k-1} \hookrightarrow \mathcal{HW}_k$$

$$V \longmapsto \left( \begin{array}{ccc} W: \mathcal{HD}_k^0 & \longrightarrow & \mathbb{R} \\ \Gamma & \longmapsto & \begin{cases} V(H_\Gamma), & \Gamma \text{ chord diagram;} \\ \sum_i V(H_{\Gamma_i}), & \Gamma \text{ trivalent diagram} \end{cases} \end{array} \right)$$

where the  $\Gamma_i$  are the chord diagram resolutions of a trivalent diagram  $\Gamma$ .

**5.3. Integrals and finite type invariants of homotopy string links.** We are now ready to state and prove the main result of this section, Theorem 5.8. This theorem states that configuration space integrals give an isomorphism between weight systems and finite type invariants of homotopy links. The way this will be shown is by exhibiting the map  $f$  above as the inverse to integration.

We first have the same statement for links, which is a combination of Theorems 4.7 and 4.11 in [27].

**Theorem 5.6.** *For  $k \geq 0$  and  $m \geq 1$ , the map*

$$I_{\mathcal{L}}^0: \mathcal{LW}_k \longrightarrow \mathcal{LV}_k/\mathcal{LV}_{k-1}$$

*given by*

$$W \longmapsto \left( L \longmapsto \sum_{\Gamma \in \mathcal{LD}_k^0} W(\Gamma)(\tilde{I}_{\mathcal{L}})_\Gamma(L) \right)$$

is an isomorphism.

*Remark 5.7.* Note that the map  $I_{\mathcal{L}}^0$  exists even for  $n > 3$ . However, one then obtains cohomology classes of  $\mathcal{L}_m^n$  in degree  $(n-3)k$  rather than in degree 0. The same is true for the map  $I_{\mathcal{H}}^0$  in Theorem 5.8 below.

We now prove the same statement for homotopy links.

**Theorem 5.8.** *For  $k \geq 0$  and  $m \geq 1$ , the map*

$$I_{\mathcal{H}}^0: \mathcal{HW}_k \longrightarrow \mathcal{HV}_k/\mathcal{HV}_{k-1}$$

*given by*

$$W \longmapsto \left( H \longmapsto \sum_{\Gamma \in \mathcal{HD}_k^0} W(\Gamma)(I_{\mathcal{H}})_{\Gamma}(H) \right)$$

*is an isomorphism.*

To make the proof of this theorem more readable, we first prove parts of it in Propositions 5.9 and 5.11.

**Proposition 5.9.** *The image of  $I_{\mathcal{H}}^0$  is a subset of  $\mathcal{HV}_k/\mathcal{HV}_{k-1}$ .*

*Proof.* First note that there is a map

$$\mathcal{HW}_k \longrightarrow \mathcal{LW}_k$$

given by extending a weight system  $W$  from  $\mathcal{HW}_k$  to  $\mathcal{LW}_k$  by  $W(\Gamma) = 0$  for  $\Gamma \in \mathcal{LD}_k \setminus \mathcal{HD}_k$ . Composing with  $I_{\mathcal{L}}^0$  gives a map

$$\mathcal{HW}_k \longrightarrow \mathcal{LV}_k/\mathcal{LV}_{k-1}$$

and this is precisely  $I_{\mathcal{H}}^0$ . But now we want to argue that the invariant produced this way is in fact locally constant on  $\mathcal{H}_m^3$  and not just on  $\mathcal{L}_m^3$ . This would show that the above map factors through  $\mathcal{HV}_k/\mathcal{HV}_{k-1}$ , i.e. that there is a commutative diagram

$$(33) \quad \begin{array}{ccc} \mathcal{HW}_k & \longrightarrow & \mathcal{HV}_k/\mathcal{HV}_{k-1} \\ \downarrow & & \downarrow \\ \mathcal{LW}_k & \xrightarrow{\cong} & \mathcal{LV}_k/\mathcal{LV}_{k-1} \end{array}$$

The right vertical map is given by restricting an invariant of homotopy links to embedded links (this is induced by the inclusion  $\mathcal{L}_m^3 \hookrightarrow \mathcal{H}_m^3$ ). We would thus get that  $I_{\mathcal{H}}^0$  produces finite type invariants of homotopy links.

Since  $I_{\mathcal{H}}^0$  is already constant on isotopic links (as it is an invariant of  $\mathcal{L}_m^3$ ), it suffices to show that this integral takes the same value on a link before and after a crossing change by the discussion at the end of Section 2.

Thus it suffices to show that given a diagram  $\Gamma \in \mathcal{HD}_k$  and links  $H^+$ ,  $H^-$  which differ only inside a ball  $B_\delta$  or radius  $\delta$  as pictured in Figure 23, we have

$$(I_{\mathcal{H}})_{\Gamma}(H^+) = (I_{\mathcal{H}})_{\Gamma}(H^-).$$

In other words,

(34)

$$\int_{\oplus_l C[\vec{d}_l + s_l; H^+, c_l(\Gamma)]} \prod_{\text{edges } (a, b) \text{ of } \Gamma} \left( \frac{x_a - x_b}{|x_a - x_b|} \right)^* \text{sym}_{S^2} - \int_{\oplus_l C[\vec{d}_l + s_l; H^-, c_l(\Gamma)]} \prod_{\text{edges } (a, b) \text{ of } \Gamma} \left( \frac{x_a - x_b}{|x_a - x_b|} \right)^* \text{sym}_{S^2} = 0$$

As usual, the configuration points  $x_a$  and  $x_b$  here correspond to diagram vertices  $a$  and  $b$ .

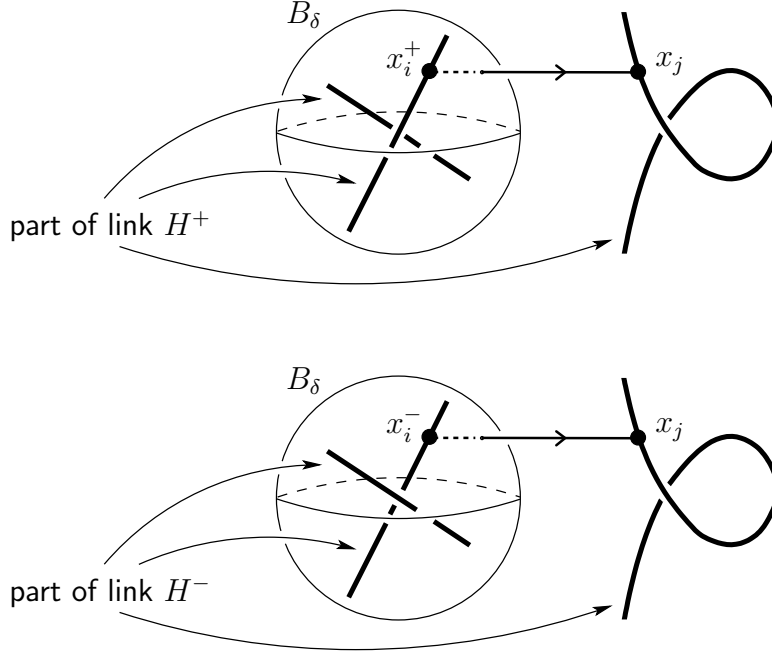


FIGURE 23. Homotopy links  $H^+$  and  $H^-$  are the same outside the ball  $B_\delta$  where they differ as pictured. The two arcs in  $B_\delta$  come from the same strand.

The domain of integration over which the two integrals differ has measure a constant times  $\delta$ , and the integrals over these regions are bounded since  $|x_a - x_b| > \epsilon > 0$  for some  $\epsilon$  independent of  $\delta$  for all  $a$  and  $b$  because such  $x_a$  and  $x_b$  will never lie on the same strand. It follows that the difference of the integrals can be made arbitrarily small.

□

*Remark 5.10.* We have used Theorem 5.6 to prove Proposition 5.9, but a direct proof is also not difficult: By construction,  $I_{\mathcal{H}}^0$  produces invariants because the restriction of the integrals to all faces cancels. Namely, recall that the differential  $d$  in  $\Omega^*(\mathcal{H}_m^3)$  is Stokes' Theorem for manifolds with corners (26), and note that the sum  $\sum W(\Gamma)(I_{\mathcal{H}})_\Gamma(H)$  breaks into triples so that we get cancellations like the one given in Figure 24. The last equality in that computation follows by the STU relation (we have omitted the labels on diagrams and signs to simplify the picture).

Similar cancellation occurs with principal faces resulting from collision of free vertices, where one now uses the IHX relation. The contributions from all principal faces thus cancel, and we



$$\begin{aligned}
 & d\left(W(\text{Y})(I_{\mathcal{H}})_{\text{Y}}(H) \pm W(\text{L})(I_{\mathcal{H}})_{\text{L}}(H) \pm W(\text{X})(I_{\mathcal{H}})_{\text{X}}(H)\right) \\
 &= W(\text{Y})d(I_{\mathcal{H}})_{\text{Y}}(H) \pm W(\text{L})d(I_{\mathcal{H}})_{\text{L}}(H) \pm W(\text{X})d(I_{\mathcal{H}})_{\text{X}}(H) \\
 &= W(\text{Y})(I_{\mathcal{H}})_{\text{Y}}(H) \pm W(\text{L})(I_{\mathcal{H}})_{\text{Y}}(H) \pm W(\text{X})(I_{\mathcal{H}})_{\text{Y}}(H) \\
 &= \left(W(\text{Y}) \pm W(\text{L}) \pm W(\text{X})\right)(I_{\mathcal{H}})_{\text{Y}}(H) \\
 &= 0
 \end{aligned}$$

FIGURE 24. Cancellation due to the STU relation

already know from Proposition 4.27 that the integrals over all other faces vanish. Thus the form  $I_{\mathcal{H}}^0(W)$  is closed for each  $W \in \mathcal{HW}_k$ . The argument to show that this is in fact a finite type  $k$  invariant is also straightforward and goes much along the lines of proofs of Proposition 5.11 and Theorem 5.8 below.

**Proposition 5.11.** *The map  $I_{\mathcal{H}}^0$  is injective.*

*Proof.* Let  $W \in \mathcal{HW}_k$  be a non-trivial weight system. Then there exists a diagram  $\Gamma \in \mathcal{HD}_k$  such that  $W(\Gamma) \neq 0$ . We may assume  $\Gamma$  is a chord diagram (i.e. with no free vertices) because otherwise we can apply the STU relation to  $\Gamma$  and resolve it as a sum of chord diagrams, and  $W$  must be non-zero on at least one of those summands. Assume the number of chords of  $\Gamma$  is equal to  $c$ , and the number of segments of  $\Gamma$  is equal to  $m$ .

We will construct a homotopy link from  $\Gamma$  much in the way we did in the discussion preceding Figure 22. To make the construction clearer we will draw a picture of  $\Gamma$ , its “horizontal representation”, with its segments all parallel horizontal line segments, numbered from top to bottom (see Figure 25)<sup>2</sup>. Start with the  $m$  parallel disjoint copies of  $\mathbb{R}$ , where the  $i$ th strand is given in standard coordinates on  $\mathbb{R}^3$  as  $\{(x, -i, 0) : x \in \mathbb{R}\}$ . In what follows, by “above” (resp. “below”) we will mean above (resp. below) the  $xy$ -plane in  $\mathbb{R}^3$ . The rough idea is to manipulate the strands to make a singular link  $H_{\Gamma}$  as follows. If there is a chord between the  $i$ th and  $j$ th strands and  $i < j$ , then strand  $i$  will pass below strands  $i+1, \dots, j-1$  intersect strand  $j$  in a single point, then pass beneath strands  $j, j-1, \dots, i+1$ , and then resume on its course along  $\{(x, -i, 0)\}$ . This is too imprecise, so we further sketch the idea below.

Define the homotopy link  $H_{\Gamma}^+$  as follows: most of  $H_{\Gamma}^+$  will lie in the  $xy$ -plane, except for crossings which take place inside small balls. If there is a chord between segment  $i$  and segment

<sup>2</sup>These are not quite the “horizontal” or “tangle” chord diagrams considered by some authors. The reason is that there could be two chords between two strands that cross and there is no way to draw all chords horizontally in such a situation. However, weight systems that are associated with Milnor invariants vanish on chord diagrams with more than one chord connecting two segments [15], so in that case one can reduce to the case of genuine tangles. More will be said about this in [20].

$j$  with  $i < j$ , then strand  $i$  goes over and then under strand  $j$ . Otherwise strand  $i$  always goes under strand  $j$ . If there is also a chord between segment  $i$  and segment  $k$ , strand  $i$  goes around strand  $k$  but before or after its linking with strand  $j$  depending of whether the latter chord is below or above the former in the horizontal representation of  $\Gamma$ .

Another way to think about this is as follows: recall how a singular link  $H_\Gamma$  can be associated to a chord diagram (see the discussion before Figure 22), and choose such a link to have the property that, away from the singularities, strand  $i$  always passes under strand  $j$  if  $i < j$ . Let  $H_\Gamma^+$  be the resolution of  $H_\Gamma$  (one of  $2^c$  many) where each singularity has been resolved so that  $i$  is the “overstrand” (we choose one of the pictures on the right side of the equation in Figure 21 depending on which strand is  $i$ ). Thus strand  $i$  passes over strand  $j$ , but because it must eventually come back to the same horizontal “level” (in the  $xy$ -plane, and with the same  $y$ -coordinate; strands have fixed behavior outside compact sets where they agree with some fixed linear embeddings of  $\mathbb{R}$  in  $\mathbb{R}^3$ ), strand  $i$  also must come back “up” under strand  $j$  at least once, or it had done so before the singularity and resolving the singularity puts it back on a level higher than strand  $j$ . Either way, there exist two consecutive crossings between strand  $i$  and  $j$  which produce a non-trivial linking. This is the only way  $i$  and  $j$  link since, if  $i$  goes under  $j$  elsewhere, then it also comes back under  $j$  and the two crossings cancel in the sense that their contributions to the integral are the same but with different signs. Also by construction, the linking between strands in  $H_\Gamma^+$  occurs precisely in the order specified by the chords of  $\Gamma$ . An example is given in Figure 25.

Now choose the form  $\text{sym}_{S^2}$  to be one that is concentrated around the north and south poles. Recall that anomalous faces are not an issue for homotopy links, so  $\text{sym}_{S^2}$  can thus be any symmetric form and need not be rotation-invariant (also see Remark 4.30). Our integrals in that case essentially count the number of times all the direction vectors prescribed by a diagram are vertical. Each time this occurs, a contribution of  $(1/2)^c$  is made to the integral.

Then  $(I_{\mathcal{H}})_\Gamma(H_\Gamma^+)$  is non-trivial, and is in fact equal to 1. This is because there are precisely  $2^c$  instances where all the direction vectors are vertical (some may be pointing up and some down) and are contributing non-trivially to the integral. For example, Figure 25 shows the case when both direction vectors are pointing up. This contributes  $1/2 \cdot 1/2 = 1/4$  to the integral. There are three other ways in which both vectors can be vertical, for the total contribution of  $4 \cdot 1/4 = 1$ .

It is also immediate that, by construction, this does not happen for any other chord diagram – because of the relative positions of chords in any other diagram, it cannot be arranged that all the direction vectors are vertical at the same time.

We thus have that

$$\sum_{\Gamma' \in \mathcal{HD}_k} W(\Gamma')(I_{\mathcal{H}})_{\Gamma'}(H_\Gamma^+) = W(\Gamma)(I_{\mathcal{H}})_\Gamma(H_\Gamma^+) = W(\Gamma) \neq 0$$

and therefore a non-trivial weight system produces a non-trivial invariant as desired. □

*Remark 5.12.* That the direction vectors contribute to the integral non-trivially only as many times as claimed is the key to the argument in the above proof. As should be clear from the proof, there are other instances when all the vectors might be vertical since each strand might

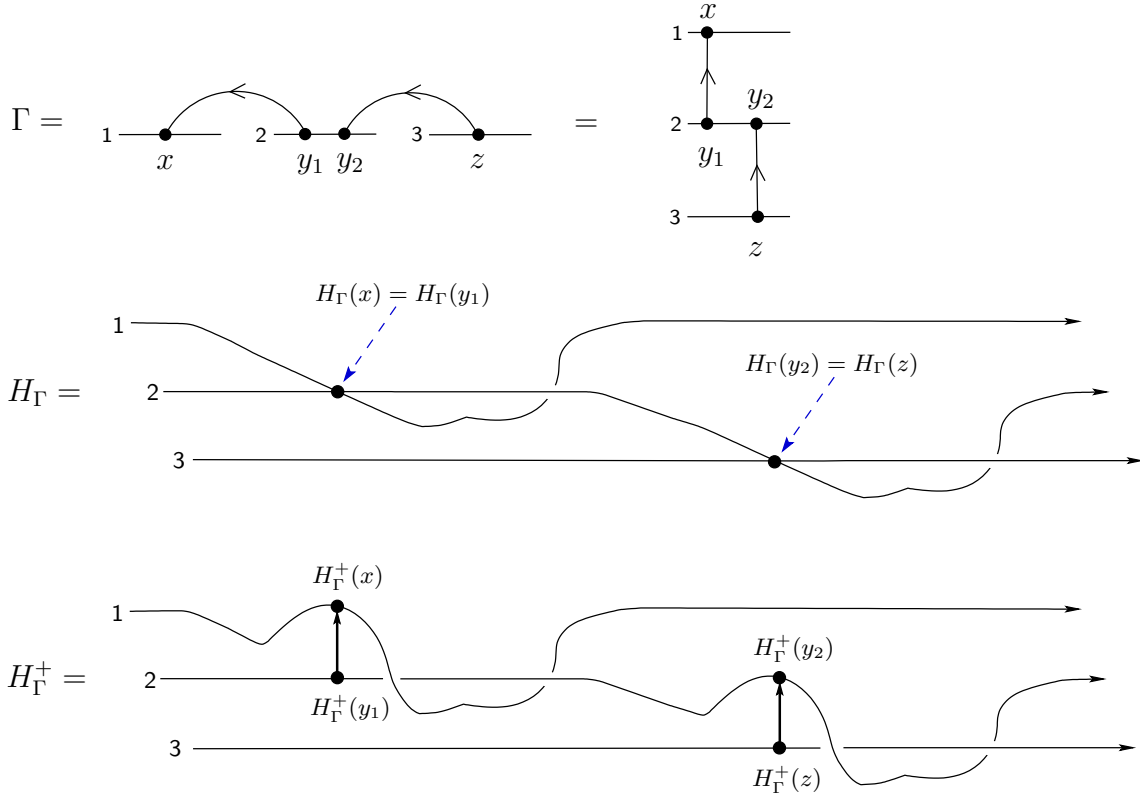


FIGURE 25. An example of a diagram  $\Gamma$ , its horizontal version, a singular homotopy link  $H_\Gamma$ , and its resolution  $H_\Gamma^+$  for which  $(I_{\mathcal{H}}^0)_\Gamma$  is non-zero.

go over or under other ones in between its non-trivial linkings prescribed by  $\Gamma$ . However, all those contributions will cancel out: if there is no chord between strand  $i$  and strand  $j$ , then strand  $i$  will always pass under strand  $j$ , and thus such crossings will not contribute to the integral.

*Proof of Theorem 5.8.* To show that  $I_{\mathcal{H}}^0$  is an isomorphism, we argue that its inverse is the map

$$f: \mathcal{HV}_k / \mathcal{HV}_{k-1} \hookrightarrow \mathcal{HW}_k$$

from (32). Because both  $f$  (essentially by definition) and  $I_{\mathcal{H}}^0$  (by Proposition 5.11) are injections, it suffices to prove that one of the compositions, say

$$(35) \quad \mathcal{HW}_k \xrightarrow{I_{\mathcal{H}}^0} \mathcal{HV}_k / \mathcal{HV}_{k-1} \xrightarrow{f} \mathcal{HW}_k,$$

is the identity.

To describe this composition explicitly, let  $\Gamma$  be a chord diagram and recall that we may choose  $H_\Gamma$  to be any singular homotopy link with (labeled) singularities as prescribed by  $\Gamma$ . We will thus choose  $H_\Gamma$  the same way as we did in Proposition 5.11, namely as an unlink with singularities where, away from the singularities, strand  $i$  always goes under strand  $j$  if  $i < j$ .

Let  $\emptyset \neq S \subset \{1, 2, \dots, k\}$  and let  $H_\Gamma^S$  be a resolution of  $H_\Gamma$  given by resolving those singularities labeled by  $S$  as in the first picture of the skein relation from Figure 21. Also let  $S^+$  be the subset of  $\{1, 2, \dots, k\}$  which indexes the resolution where strand  $i$  is chosen to go over strand  $j$  for all  $1 \leq i < j \leq m$ . Thus  $H_\Gamma^+$  that was used in the proof of Proposition 5.11 is precisely  $H_\Gamma^{S^+}$  in this notation. The composition (35) is then given by

$$W(\Gamma) \mapsto \sum_{\emptyset \neq S \subset \{1, 2, \dots, k\}} \sum_{\Gamma' \in \mathcal{HD}_k} W(\Gamma')(I_{\mathcal{H}})_{\Gamma'}(H_\Gamma^S).$$

This will be the identity if we can show

$$(36) \quad (I_{\mathcal{H}})_{\Gamma'}(H_\Gamma^S) = \begin{cases} 1, & \Gamma' = \Gamma \text{ and } S = S^+; \\ 0, & \Gamma' \neq \Gamma \text{ or } S \neq S^+. \end{cases}$$

As in the proof of Proposition 5.11, we again choose the form  $\text{sym}_{S^2}$  to be one that is concentrated around the poles. With this in mind, we have:

- When  $\Gamma' = \Gamma$  and  $S = S^+$ ,  $(I_{\mathcal{H}})_{\Gamma}(H_\Gamma^{S^+}) = 1$ . The argument is the same as in Proposition 5.11; in this case the direction vectors detect linking at the same time;
- It is also clear that, for any other resolution of  $H_\Gamma$  (i.e. when  $S \neq S^+$ ), the detection from the previous case will not occur. Namely, if a singularity between strand  $i$  and  $j$  is resolved so that  $i$  is the understrand, then the direction vector associated to the chord, which is in turn associated to the singularity, will produce no contribution to the integral;
- Similarly, for any chord diagram different from  $\Gamma$  (i.e. if  $\Gamma' \neq \Gamma$ ), the direction vectors cannot all be vertical at the same time (this is again an argument from Proposition 5.11);
- Lastly, if  $\Gamma'$  is a diagram with free vertices, then the direction vectors again cannot all be vertical. This is because each resolution of  $H_\Gamma$  is “almost planar”, namely it lies in the plane except for the crossings which are contained in small disjoint balls. Therefore, since free vertices are trivalent and each has a path to a segment vertex, it follows that at least one vector pointing to or from a free configuration point is not vertical.

This proves (36) and completes the proof of the theorem.  $\square$

*Remark 5.13.* Even though in the proof of Proposition 5.9 we appealed to Theorem 5.6 and the fact that  $I_{\mathcal{L}}^0$  is a universal finite type invariant of ordinary string links, it is easy to prove Proposition 5.9 in a way that is independent of Theorem 5.6, as mentioned in Remark 5.10. In addition, the proof of the latter two statements essentially works the same way for string links as it does for homotopy string links. In light of the fact that the proof of Theorem 5.6 is only outlined in [27], one can thus regard the complete picture given here for finite type invariants of homotopy string links as also giving a fairly complete picture of finite type invariants for ordinary string links.

**5.4. Milnor invariants of homotopy string links.** With Theorem 5.8 in hand, we can now quickly deduce the corollary about Milnor invariants of string links as promised in the Introduction.

For  $m$ -component string links, each non-repeating index Milnor invariant  $\mu_{i_1 i_2 \dots i_{k+1}}$ ,  $1 \leq i_j \leq m$ , is well-defined (for closed links, there is an indeterminacy, modulo which one gets the  $\overline{\mu}$

invariants), and it is a finite type  $k$  invariant [3, 13]. Furthermore, this is a link-homotopy invariant [17]. Thus  $\mu_{i_1 i_2 \dots i_{k+1}}$  can be thought of as a finite type invariant of  $\mathcal{H}_m^3$  (here we again use the discussion following Corollary 2.4).

We have then the following consequence of Theorem 5.8.

**Theorem 5.14.** *Each Milnor invariant  $\mu_{i_1 i_2 \dots i_{k+1}}$  of string links of  $m$  components is given by*

$$(37) \quad \mu_{i_1 i_2 \dots i_{k+1}}(H) = (I_{\mathcal{H}}^0(W))(H) = \sum_{\Gamma \in \mathcal{HD}_k} W(\Gamma)(I_{\mathcal{H}})_{\Gamma}(H)$$

for some weight system  $W \in \mathcal{HW}_k$ .

We can refine this statement. If  $k + 1 < m$ , then some index  $j$  between 1 and  $m$  does not appear in the subscript of  $\mu_{i_1 i_2 \dots i_{k+1}}$ , and we then have a Milnor invariant of  $(m-1)$ -component links, namely an invariant of the link obtained by deleting the  $j$ th strand. By relabeling, we can assume that that the deleted strand is in fact the  $m$ th one. To understand Milnor invariants, it suffices to study those invariants of  $m$ -component links that are not induced by the projection

$$\mathcal{H}_m^3 \longrightarrow \mathcal{H}_{m-1}^3$$

given by deleting the  $m$ th strand of a link. This means that, in the sum from (37), we only take those diagrams  $\Gamma$  with segment vertices appearing on *all* segments. If the sum is taken over only those diagrams that do not have any segment vertices on, say, the  $m$ th segment, then one obtains an invariant of  $(m-1)$ -component links. This is easy to see as the such diagrams account for all the necessary cancellations of integration along faces and thus produce a closed form. We will call diagrams with segment vertices on all segments *maximal* and will denote them by  $\Gamma_{\max}$ .

It follows that, since  $\mu_{i_1 i_2 \dots i_m}$  is a type  $m-1$  invariant, each  $\Gamma_{\max}$  must have  $2(m-1)$  vertices, at least  $m$  of which are segment vertices, lying on  $m$  segments. These can also be characterized at forests with at least  $m$  but no more than  $2(m-1)$  leaves with  $m$  distinct labels (each label is associated with a unique segment/strand). Recall that by a forest we mean a disjoint union of trees, and by a tree we mean the collection of vertices and edges, but not segments, of a diagram, where the leaves are the segment vertices.

We thus get the following

**Corollary 5.15.** *Each Milnor invariant  $\mu_{i_1 i_2 \dots i_m}$  of string links of  $m$  components is given by*

$$(38) \quad \mu_{i_1 i_2 \dots i_m}(H) = (I_{\mathcal{H}}^0(W))(H) = \sum_{\Gamma_{\max} \in \mathcal{HD}_{m-1}} W(\Gamma)(I_{\mathcal{H}})_{\Gamma_{\max}}(H)$$

for some weight system  $W \in \mathcal{HW}_{m-1}$ .

*Remark 5.16.* Suppose that in addition we required that  $\Gamma_{\max} \in \mathcal{HD}_{m-1}$  be connected. It is immediate that such a trivalent diagram must have precisely  $m$  segment vertices (one on each of the  $m$  segments) and  $m-2$  free vertices. Since diagrams in  $\mathcal{HD}_m$  have no loops of edges, it follows that a connected  $\Gamma_{\max}$  is precisely a tree with  $m$  leaves. One consequence, which will be elaborated on in [20], is that such diagrams are in one-to-one correspondence, via configuration space integrals, with Milnor invariants  $\bar{\mu}_{i_1 i_2 \dots i_m}$  of *closed* links. This is because Milnor weight systems have to satisfy more relations in the case of closed links, and so fewer diagrams are needed to produce them.

The next step, which the authors plan to carry out in [20], is to understand precisely which weight systems appear in Corollary 5.15. In particular, the authors plan to use the combinatorial properties of such “Milnor weight systems” established in [15]. We will also explore the connection to [10]; one of the results of that paper is that Milnor invariants of string links correspond to the tree part of the Kontsevich integral, and it is this integral that gives an alternative way of showing that weight systems correspond to finite type invariants (in fact, the Kontsevich integral provided the first proof of this theorem). In addition, the authors will undertake further study of configuration space integrals and Milnor invariants in the context of manifold calculus of functors in [22].

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